Approximate convexity and submonotonicity

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Abstract It is shown that a locally Lipschitz function is approximately convex if, and only if, its Clarke subdifferential is a submonotone operator. Consequently, in finite dimensions, the class of locally Lipschitz approximately convex functions coincides with the class of lower-C^1 functions. Directional approximate convexity is introduced and shown to be a natural extension of the class of lower-C^1 functions in infinite dimensions. The following characterization is established: a multivalued operator is maximal cyclically submonotone if, and only if, it coincides with the Clarke subdifferential of a locally Lipschitz directionally approximately convex function, which is unique up to a constant. Furthermore, it is shown that in Asplund spaces, every regular function is generically approximately convex.

1 Introduction

A locally Lipschitz function $f : U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}^n$, is called *lower-C^1*, if for every $x_0 \in U$, there exist a neighborhood V of x_0, a compact set S and a jointly continuous function $g : V \times S \to \mathbb{R}$, such that for all $x \in V$ we have $f(x) = \max_{s \in S} g(x, s)$ and the derivative $D_x g$ (exists and) is jointly continuous.

The above class of functions has been introduced by Spingarn in [23]. There, it has been shown that a locally Lipschitz function $f : U \to \mathbb{R}$ is lower-C^1 if, and only if, its Clarke subdifferential ∂f is *submonotone*[1] at every $x \in U$, a multivalued operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ being here called submonotone at $x_0 \in X$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\|, \tag{1}$$

for all $x_i \in B(x_0, \delta)$ and all $x_i^* \in T(x_i)$, $i = 1, 2$. We shall adopt the same definition for a multivalued operator $T : X \rightrightarrows X^*$ from a Banach space X to its dual X^*.

Recently Ngai, Luc & Thera [16] introduced and studied the class of approximately convex functions f defined on a Banach space X. Let us recall their definition.

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[1] strictly submonotone according to [23].

Definition 1 *A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called approximately convex at $x_0 \in X$, if for every $\varepsilon > 0$ there exists $\delta > 0$ (depending on x_0 and ε) such that for all $x, y \in B(x_0, \delta)$ and $t \in (0, 1)$*

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + \varepsilon t(1 - t)\|x - y\|. \qquad (2)$$

Furthermore, we say that f is approximately convex (respectively, generically approximately convex), if it is approximately convex at every $x_0 \in X$ (respectively, at every x_0 in a G_δ dense set).

Locally Lipschitz approximately convex functions are regular, while the converse is in general false (Remark 6). However, in Asplund spaces, regular locally Lipschitz functions are generically approximately convex (Proposition 5). We shall also show that a locally Lipschitz function f is approximately convex if, and only if, its Clarke subdifferential ∂f is submonotone. Consequently, in finite dimensions, a locally Lipschitz function is lower-C^1 if and only if it is approximately convex. The latter states the equivalence of a pure analytic definition (lower-C^1 function) with a geometric one (first order relaxation of convexity).

Let us mention that a result of such type is known to hold for the smaller class of lower-C^2 functions ([22]), that is, in finite dimensions, lower-C^2 functions are exactly the locally Lipschitz weakly convex functions (second order relaxation of convexity). Moreover, every such function is characterized by its (locally) decomposability as a sum of a convex continuous and a concave quadratic function, see [24] e.g. See also [10], [18], [4], [20] and references therein for related topics. Recently, Zs. Páles showed that ε-approximately convex functions on the real line are decomposable into a sum of a convex and a Lipschitz function ([17, Theorem 5]), however the existence of such a characterization in higher dimensions remains open. In fact, the results of [17] deal with the more general notion of (ε, δ)-approximate convexity, which extends also the notion of approximate convexity considered in the pioneering works on this subject, [11] and [9]. We also quote the recent relevant works [6] and [13].

Directional approximate convexity – introduced in Section 3 – is shown to be a natural extension for the notion of lower-C^1 functions in infinite dimensions: a locally Lipschitz function f is *directionally approximately convex* if, and only if, ∂f is *directionally submonotone* (see Definition 7; this class of operators has been previously defined in [8, Definition 1.2] under the name 'strictly submonotone mappings'; subsequently this notion was called in [5] 'directional strict submonotonicity' or 'ds-submonotonicity'). In finite dimensions, directional approximate convexity and approximate convexity coincide. Speaking of operators, the same is true for submonotonicity and directional submonotonicity. Combining with results from [5] we shall thus conclude that the class of *maximal cyclically submonotone* operators – notion introduced in [12] in finite dimensions, and extended into infinite dimensions in [5, Definition 6] – coincides with the class of Clarke subdifferentials of locally Lipschitz directionally approximately convex functions. This result is analogous to a classical result of Rockafellar asserting the coincidence of the class of maximal cyclically monotone

operators with the subdifferentials of lower semicontinuous convex functions. Recent literature contains various other interesting extensions of this latter result through different approaches, see [19], [2] e.g. See also [1] and references therein.

2 Characterization of approximate convexity

In the sequel, let $B(x, \delta)$ stand for the open ball centered at $x \in X$ with radius $\delta > 0$ and let S_X denote the unit sphere of X. If $f : X \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function with domain

$$\text{dom} f := \{x \in X : f(x) \neq +\infty\},$$

the *Clarke-Rockafellar subdifferential* of f at $x_0 \in \text{dom} f$ is defined by

$$\partial^{\uparrow} f(x_0) = \{x^* \in X : \langle x^*, u \rangle \leq f^{\uparrow}(x_0, u), \forall u \in X\},$$

where

$$f^{\uparrow}(x_0, u) = \sup_{\varepsilon > 0} \limsup_{\substack{x \to_f x_0 \\ t \searrow 0^+}} \inf_{v \in B(u, \varepsilon)} \frac{f(x + tv) - f(x)}{t}.$$

(In the above formula $t \searrow 0^+$ indicates the fact that $t > 0$ and $t \to 0$, and $x \to_f x_0$ means that both $x \to x_0$ and $f(x) \to f(x_0)$.)

Whenever f is locally Lipschitz we have $f^{\uparrow}(x_0, u) = f^0(x_0, u)$ for all $u \in X$, so that $\partial^{\uparrow} f(x_0) = \partial f(x_0)$, where

$$f^o(x; u) = \limsup_{(y, t) \to (x, 0^+)} \frac{f(y + tu) - f(y)}{t}$$

is the Clarke derivative of f at the direction u and

$$\partial f(x) = \{x^* \in X^* : \langle x^*, u \rangle \leq f^o(x, u), \forall u \in X\}$$

is the Clarke subdifferential of f at x.

Theorem 2 *Let f be locally Lipschitz on X and $x_0 \in X$. The following are equivalent:*

(i) f is approximately convex at x_0.

(ii) For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in B(x_0, \delta)$ and $x^* \in \partial f(x)$

$$f(x + u) - f(x) \geq \langle x^*, u \rangle - \varepsilon \|u\|, \tag{3}$$

whenever $\|u\| < \delta$ is such that $x + u \in B(x_0, \delta)$.

(iii) ∂f is submonotone at x_0.

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Note that (ii) says that Clarke subdifferentials around x_0 are (uniformly) local ε-supports in the sense of Ekeland and Lebourg ([7]).

Proof. Implication (i)$\Longrightarrow$(ii) follows from [15, Proposition 4.3] and the definition of approximate convexity (Definition 1). (This implication is also valid for lower semicontinuous functions.)

To prove (ii)$\Longrightarrow$(iii), let $\varepsilon > 0$ and take $\delta > 0$ such that relation (3) holds for $\varepsilon/2 > 0$ and for all $x \in B(x_0, \delta)$. Then for x, y in $B(x_0, \frac{\delta}{2})$, $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$ we have

$$f(y) - f(x) \geq \langle x^*, y - x \rangle - \frac{\varepsilon}{2}\|y - x\|$$

and

$$f(x) - f(y) \geq \langle y^*, x - y \rangle - \frac{\varepsilon}{2}\|y - x\|.$$

Adding the above inequalities we obtain (1), that is ∂f is submonotone at x_0.

Finally for the implication (iii)$\Longrightarrow$(i), let $\varepsilon > 0$ and take $\delta > 0$ such that relation (1) holds. Let us consider any x, y in $B(x_0, \delta)$ and $t \in (0, 1)$ and let us set $x_t = tx + (1 - t)y$. Then applying Lebourg's Mean Value theorem ([14, Theorem 1.7]) on the segment $[x, x_t]$ we obtain a point $z_1 \in [x, x_t[$ such that for some $z_1^* \in \partial f(z_1)$

$$\langle z_1^*, x_t - x \rangle = f(x_t) - f(x). \tag{4}$$

Similarly, there exists a point $z_2 \in [y, x_t[$ such that for some $z_2^* \in \partial f(z_2)$

$$\langle z_2^*, x_t - y \rangle = f(x_t) - f(y). \tag{5}$$

Since $x_t - x = (1 - t)(y - x)$ and $x_t - y = t(x - y)$, multiplying relations (4) and (5) by t and $(1 - t)$ respectively, and adding the resulting equalities we obtain

$$tf(x) + (1 - t)f(y) - f(x_t) = t(1 - t)\langle z_1^* - z_2^*, x - y \rangle.$$

Since

$$\frac{x - y}{\|x - y\|} = \frac{z_1 - z_2}{\|z_1 - z_2\|},$$

using (1) we obtain (2), that is f is approximately convex at x_0. $\qquad\square$

It follows from the above statement that a locally Lipschitz function f is approximately convex if, and only if, ∂f is submonotone. This yields directly the following corollary.

Corollary 3 *In finite dimensions a locally Lipschitz function is approximately convex if, and only if, it is lower-C^1.*

We shall further establish a generic result concerning regular functions in Asplund spaces. We first need the following result stating that generic approximate convexity (Definition 1) is separably determined. The proof is based on a separable reduction argument.

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Proposition 4 *A function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *is generically approximately convex, if for every closed separable subspace* Y *of* X *the restriction* $f \mid_Y$ *is generically approximately convex in* Y.

Proof For any closed subspace Z of X and any $p \geq 1$, let $U_p(Z)$ be the set of all $z \in Z$, for which there exists $\delta > 0$ such that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \frac{1}{p}t(1-t)\|x-y\|,$$

for all $x, y \in B(z, \delta) \cap Z$ and for all $t \in [0, 1]$. Obviously $U_p(Z)$ is (relatively) open in Z.

Let us suppose that the restriction of the function f to any separable subspace is generically approximately convex. In order to conclude that f is generically approximately convex, it clearly suffices to show that $U_p(X)$ is dense in X for all $p \geq 1$.

Let us suppose, towards a contradiction, that this is not the case for some $p_0 \geq 1$. Then there exists a nonempty open subset U of X such that $U \cap U_{p_0}(X) = \emptyset$.

Pick any $z_1 \in U$. Then for every $n \geq 1$, there exist $x_1^n, y_1^n \in B(z_1, 1/n)$ and $t_1^n \in (0, 1)$ such that

$$f(t_1^n\, x_1^n + (1-t_1^n)\, y_1^n) > \quad t_1^n\, f(x_1^n) + (1-t_1^n)\, f(y_1^n) +$$

$$+ \tfrac{1}{p_0}\, t_1^n\, (1-t_1^n)\|x_1^n - y_1^n\|.$$

Let Z_1 be the closed (separable) subspace generated by the sequences $\{x_1^n\}_n$, $\{y_1^n\}_n$ and the point z_1, and set $U_1 = U \cap Z_1$. Let $\{z_{2,k}\}_{k \geq 1}$ be a dense subset of U_1. Then, for every $k \geq 1$ and $n \geq 1$, there exist $x_{2,k}^n, y_{2,k}^n \in B(z_{2,k}, 1/n)$ and $t_2^n \in (0, 1)$ such that

$$f(t_2^n\, x_{2,k}^n + (1-t_2^n)\, y_{2,k}^n) > \quad t_2^n\, f(x_{2,k}^n) + (1-t_2^n)\, f(y_{2,k}^n) +$$

$$\tfrac{1}{p_0}\, t_2^n\, (1-t_2^n)\|x_{2,k}^n - y_{2,k}^n\|.$$

Let Z_2 be the closed subspace generated by the space Z_1 and the sequences $\{x_{2,k}^n\}_{k,n}$, $\{y_{2,k}^n\}_{k,n}$ and $\{z_{2,k}\}_{k \geq 1}$. Set $U_2 = U \cap Z_2$. Proceeding like this, we obtain an increasing sequence of separable subspaces $\{Z_s\}_{s \geq 1}$ of X.

Set $Z_\infty = \overline{\bigcup_{s \geq 1} Z_s}$ and $U_\infty = U \cap Z_\infty$. We claim that $U_\infty \backslash U_{p_0}(Z_\infty)$ is dense in U_∞.

Indeed, for every $u \in U_\infty$ and $\varepsilon > 0$, there exists $s \geq 1$ and $u_s \in U \cap Z_s := U_s \subset U_\infty$ such that $\|u_s - u\| < \varepsilon/2$. The above construction now shows that there exists $v_s \in U_s \backslash U_{p_0}(Z_\infty)$ such that $\|v_s - u_s\| < \varepsilon/2$. It follows that $v_s \in B(u, \varepsilon) \cap [U_\infty \backslash U_{p_0}(Z_\infty)]$, hence $U_\infty \backslash U_{p_0}(Z_\infty)$ is dense in U_∞.

Since both sets U_∞ and $U_{p_0}(Z_\infty)$ are relatively open in Z_∞, we conclude that $U_\infty \cap U_{p_0}(Z_\infty) = \emptyset$. This clearly contradicts the fact that the restriction $f\mid_{Z_\infty}$ of f into the separable space Z_∞ is generically approximately convex. $\square$

Proposition 5 *Every regular function ([3, Definition 2.3.4]) in Asplund spaces is generically approximately convex.*

Proof. Suppose first that X is separable. Then from [21, Theorem 2.8] the (minimal w*-cusco) operator ∂f is generically single-valued and $(\|.\|, \|.\|)$-continuous. It follows that ∂f is submonotone at every point x_0 in which $\partial f(x_0)$ is singleton and ∂f is $(\|.\|, \|.\|)$-continuous. Thus, by Theorem 2(iii)$\Longrightarrow$(i) we conclude that f is generically approximately convex. The general case follows from Proposition 4 (separable reduction argument). $\square$

Remark 6 *(i) Regular functions are not approximately convex in general: Spingarn ([23, page 84]) gives an example of a regular function in $\mathbb{R}^2$ whose Clarke subdifferential is not submonotone at some point, and therefore the function is not approximately convex.*

(ii) It follows from Theorem 2 [(i)$\Longleftrightarrow$(iii)] and [8, Theorem 4.1] that every approximately convex function is regular. This was also proved in [16, Theorem 3.6]. In fact, the same is true for the class of directionally approximately convex functions, see Definition 8 below.

3 Directional approximate convexity and cyclic submonotonicity

Let us recall from [8, Definition 1.2] (see also [5]) the following definition.

Definition 7 *A multivalued operator $T : X \rightrightarrows X^*$ is called directionally submonotone, if for any $x_0 \in X$, $\varepsilon > 0$ and $e \in S_X$ there exists $\delta > 0$ such that*

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\|, \tag{6}$$

whenever $x_1, x_2 \in B(x_0, \delta)$, $x_i^* \in T(x_i)$ $(i = 1, 2)$ and

$$\frac{x_1 - x_2}{\|x_1 - x_2\|} \in B(e, \delta). \tag{7}$$

Comparing with the definition of submonotonicity in relation (1), the above definition imposes the additional (directional) constraint (7). In the same spirit we introduce the following notion of directional approximate convexity.

Definition 8 *A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called directionally approximately convex, if for any $x_0 \in X$, $e \in S_X$ and $\varepsilon > 0$ there exists $\delta = \delta(x_0, e, \varepsilon) > 0$ such that for $x, y \in B(x_0, \delta)$ and $t \in (0, 1)$ we have*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon t(1-t)\|x - y\|, \tag{8}$$

provided that

$$\frac{x - y}{\|x - y\|} \in B(e, \delta).$$

In finite dimensions, thanks to the compactness of the unit sphere S_X it follows by a standard argument that approximate convexity and directional approximate convexity (respectively, submonotonicity and directional submonotonicity) coincide.

We shall further need the following lemma. The proof borrows heavily from techniques employed in [15], [16].

Lemma 9 *Let f be a lower semicontinuous directionally approximately convex function. Then for every $x_0 \in X$, $e \in S_X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in B(x_0, \delta)$ and all $t \in (0, 1)$ satisfying $x + te \in B(x_0, \delta)$ we have*

$$\frac{f(x + te) - f(x)}{t} \geq f^\uparrow(x; e) - \varepsilon.$$

Proof. Let $x_0 \in X$, $e \in S_X$ and $\varepsilon > 0$ and take $\delta = \delta(x_0, e, \varepsilon) > 0$ as in Definition 8 so that relation (8) is satisfied. Let $x \in B(x_0, \delta/3)$ and choose $t \neq 0$ such that for $u = te$ we have $x + u \in B(x_0, \delta/3)$ and $\|u\| < \delta/3$. Then for all $y \in B(x_0, \delta/3)$, $v \in X$ with $0 < \|v\| < 2\delta/3$ and $(v/\|v\|) \in B(e, \delta)$ and $s \in (0, 1)$ we obtain applying (8) that

$$f(y + sv) = f((1-s)y + s(y + v)) \leq sf(y + v) + (1-s)f(y) + \varepsilon s(1-s)\|v\|,$$

whence

$$\frac{f(y + sv) - f(y)}{s} \leq f(y + v) - f(y) + \varepsilon(1-s)\|v\|.$$

In particular for $v = u + x - y$ we have $v \in B(u, \delta)$ and consequently

$$\inf_{v \in B(u, \delta)} \frac{f(y + sv) - f(y)}{s} \leq f(x + u) - f(y) + \varepsilon(1-s)\|u + x - y\|.$$

It follows that

$$\limsup_{\substack{y \to_f x \\ s \searrow 0^+}} \inf_{v \in B(u, \delta)} \frac{f(y + sv) - f(y)}{s} \leq f(x + u) - f(x) + \varepsilon\|u\|.$$

Since the above is valid for all $\delta > 0$ it follows that

$$f^\uparrow(x; u) \leq f(x + u) - f(x) + \varepsilon\|u\|.$$

Since $\|u\| = t$, the assertion follows. $\qquad\qquad\qquad\qquad\qquad\qquad$ □

We shall now recall from [5, Definition 6] the notion of cyclic submonotonicity. This definition requires previously the notion of δ-subdivision of a closed polygonal path. (We call *closed polygonal path* a finite family of points $\{w_h\}_{h=1}^m$ where $m > 1$ and $w_1 = w_m$ and we denote by $[w_h]_{h=1}^m$, the union of the consecutive segments $[w_h, w_{h+1}]$ for $h = 1, \ldots m - 1$.)

Definition 10 *Given $\delta > 0$, we say that $\{x_i\}_{i=1}^n$ is a δ-subdivision of the closed polygonal path $[w_h]_{h=1}^m$, if*

(i) $\{x_i\}_{i=1}^n \subseteq B_\delta([w_h]_{h=1}^m)$

(ii) $x_n = x_1$ and $\|x_{i+1} - x_i\| < \delta$, for $i = 1, 2, \ldots n - 1$.

(iii) there exists a finite sequence $\{k_h\}_{h=1}^m$ with $1 = k_1 < k_2 < \ldots < k_m := n$ such that for $1 \leq h \leq m - 1$ we have:

$$k_h \leq i < k_{h+1} \implies \|\frac{x_{i+1} - x_i}{\|x_{i+1} - x_i\|} - \frac{w_{h+1} - w_h}{\|w_{h+1} - w_h\|}\| < \delta.$$

We are now ready to give the following definition.

Definition 11 *An operator $T : X \rightrightarrows X^*$ is called cyclically submonotone, if for any closed polygonal path $[w_h]_{h=1}^m$ and $\varepsilon > 0$, there exists $\delta > 0$, such that for all δ-subdivisions $\{x_i\}_{i=1}^n$ of $[w_h]_{h=1}^m$ and all $x_i^* \in T(x_i)$ one has*

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \leq \varepsilon \sum_{i=1}^{n-1} \|x_{i+1} - x_i\|. \qquad (9)$$

If U is an open subset of X, an operator T is said to be *cyclically submonotone on U* if (9) holds for closed polygonal paths and δ-subdivisions in U. Furthermore, a cyclically submonotone operator T on U is called *maximal cyclically submonotone* on U, if there is no cyclically submonotone operator $S \neq T$ such that $T(x) \subseteq S(x)$ for all $x \in U$.

The forthcoming result has been established in [5]; it extends the finite dimensional case announced in [12].

Theorem 12 *If U is an open connected subset of a Banach space X, then:*

(i) Every cyclically submonotone operator on U is directionally submonotone on U.

(ii) If f is a locally Lipschitz function and ∂f is directionally submonotone, then ∂f is also maximal cyclically submonotone.

(iii) Every maximal cyclically submonotone operator T on U coincides with the Clarke subdifferential ∂f of a unique (up to a constant) locally Lipschitz function f.

If $X = \mathbb{R}^n$, the class of locally Lipschitz functions arising in (ii) and (iii) of Theorem 12 is exactly the class of lower-C^1 functions (or equivalently, the class of approximately convex functions). The following result shows that directional approximate convexity is an appropriate extension of the latter class in infinite dimensions.

Theorem 13 *For a locally Lipschitz function f on a Banach space X the following are equivalent:*

(i) f is directionally approximately convex.

(ii) For every $x_0 \in X$, $e \in S_X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in B(x_0, \delta)$ and $x^* \in \partial f(x)$

$$f(x+u) - f(x) \geq \langle x^*, u \rangle - \varepsilon \|u\|$$

whenever $\|u\| < \delta$, $x + u \in B(x_0, \delta)$ and $\frac{u}{\|u\|} \in B(e, \delta)$.

(iii) ∂f is directionally submonotone.

(iv) ∂f is maximal cyclically submonotone.

Proof. (i)$\Longrightarrow$(ii). It follows easily from Lemma 9 and the fact that if f is locally Lipschitz, the function $u \longmapsto f^o(x; u) = f^\uparrow(x; u)$ is upper semicontinuous.

(ii)$\Longrightarrow$(iii). For any $x_0 \in X$, $e \in S_X$ and $\varepsilon > 0$, let $\delta > 0$ be such that for all $z \in B(x_0, \delta)$ and all $z^* \in \partial f(z)$ we have

$$f(z + tu) - f(z) \geq \langle z^*, u \rangle - \frac{\varepsilon}{2} \|u\|$$

whenever $\|u\| < \delta$ is such that $z + u \in B(x_0, \delta)$ and $\frac{u}{\|u\|} \in B(e, \delta) \cup B(-e, \delta)$.

Let now any x, y in $B(x_0, \frac{\delta}{2})$ satisfying $\frac{y-x}{\|y-x\|} \in B(e, \delta)$. Applying the previous formula (for $z = x$ and $u = y - x$) we obtain for all $x^* \in \partial f(x)$

$$\langle x^*, y - x \rangle \leq f(y) - f(x) + \frac{\varepsilon}{2} \|x - y\|.$$

Similarly, (for $z = y$ and $-u = x - y$) we get

$$\langle -y^*, y - x \rangle \leq f(x) - f(y) + \frac{\varepsilon}{2} \|x - y\|.$$

Adding the above inequalities we obtain (6). Thus ∂f is directionally submonotone.

(iii)$\Longrightarrow$(i) Let $x_0 \in X$, $e \in S_X$, $\varepsilon > 0$ and take $\delta > 0$ such that relation (6) holds. Let us consider any x, y in $B(x_0, \delta)$ such that $\frac{y-x}{\|y-x\|} \in B(e, \delta)$ and any $t \in (0, 1)$ and let us set $x_t = tx + (1-t)y$. Then applying Lebourg's Mean Value theorem on the segments $[x, x_t]$ and $[y, x_t]$ we obtain points $z_1 \in [x, x_t[$ and $z_2 \in [y, x_t[$ such that for some $z_1^* \in \partial f(z_1)$ and $z_2^* \in \partial f(z_2)$ we have

$$\langle z_1^*, x_t - x \rangle = f(x_t) - f(x)$$

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and

$$\langle z_2^*, x_t - y \rangle = f(x_t) - f(y).$$

Since $x_t - x = (1-t)(y-x)$ and $x_t - y = t(x-y)$, multiplying the first relation above by t and the second by $(1-t)$ and adding the resulting equalities we obtain

$$tf(x) + (1-t)f(y) - f(x_t) = t(1-t)\langle z_1^* - z_2^*, x - y \rangle.$$

Since

$$\frac{x-y}{\|x-y\|} = \frac{z_1 - z_2}{\|z_1 - z_2\|} \in B(e, \delta)$$

using (6) we obtain (8), that is f is directionally approximately convex.

Finally, the equivalence (iii)$\Longleftrightarrow$(iv) follows from Theorem 12. $\qquad\square$

Examples
1. ([5, Proposition 19]) The function $f(x) = -d_A(x)$ is directionally approximately convex, whenever X has a uniformly Gâteaux differentiable norm and $d_A(.)$ is the distance function (generated by this norm) of the nonempty closed subset A of X.

2. ([5, Proposition 20]) The composition $g \circ F$ of an approximately convex function $g : Y \to \mathbb{R}$ with a strictly Gâteaux differentiable function $F : X \to Y$ is approximately convex.

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