

A DUAL CHARACTERIZATION OF THE RADON-NIKODYM
PROPERTY

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Abstract We prove that a Banach space X has the Radon-Nikodym property if, and only if, every weak*-lower semicontinuous convex continuous function f of X^* is Gâteaux differentiable at some point of its domain with derivative in the predual space X .

Key words Radon-Nikodym property, Gâteaux derivative, weakly exposed point, convex function.

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1 Introduction

Collier [5] has shown that a Banach space X has the Radon-Nikodym property if, and only if, all weak*-lower semicontinuous convex continuous functions on the dual space X^* are generically Fréchet differentiable. (Such a dual space was called in [5] weak*-Asplund). In this article we give the following characterization of the Radon-Nikodym property in terms of Gâteaux derivatives.

Theorem 1 *A Banach space X has the Radon-Nikodym property if, and only if, every weak*-lower semicontinuous convex continuous function on X^* is Gâteaux differentiable at some point of its domain with derivative in the predual space X .*

Since Fréchet derivatives of weak*-lower semicontinuous convex continuous functions of X^* are always elements of X (see [7] e.g.), the improvement upon the aforementioned result of Collier consists on replacing the Fréchet derivative by Gâteaux and on passing from a dense differentiability assumption to the existence of the derivative at one point.

If X does not have the Radon-Nikodym property, then it is possible to have nowhere Fréchet differentiable weak*-lower semicontinuous convex continuous functions on X^* for which the set of points

where the Gâteaux derivative exists and belongs to the predual space is dense (see Proposition 8). Concurrently, it is also possible to have weak*-lower semicontinuous convex continuous functions on X^* that are generically Gâteaux differentiable with all derivatives in $X^{**} \setminus X$. Indeed, consider the Banach space $X = c_0(\mathbb{N})$, its dual space $X^* = \ell^1(\mathbb{N})$ and the function $g(x) = \|x\|_1$, see [10, Example 1.4 (b)] for details.

Let us finally note that characterizations of the Radon-Nikodym property for dual Banach spaces in terms of the Gâteaux derivative are recently established by Giles in [8, Theorem 2].

The proof of Theorem 1 is given in Section 3, while in the following section we fix our notation and we recall relevant definitions.

2 Preliminaries

In the sequel, $(X, \|\cdot\|)$ will be a Banach space and $(X^*, \|\cdot\|)$ will be its dual. We denote by B_X the closed unit ball of X and by \mathbb{R} (resp. \mathbb{N}) the set of all real (resp. positive integer) numbers. For any $x \in X$ and any $p \in X^*$ we denote by $\langle p, x \rangle$ the value of the functional p at the point x . Similarly, for any z^{**} in X^{**} we denote by $\langle p, z^{**} \rangle$ the value of z^{**} at p . We also denote by $\overline{\text{co}}F$ the closed convex hull of the set F . For any non-empty closed bounded subset F of X we denote by ψ_F the indicator function of F ($\psi_F(x) := 0$, if $x \in F$ and $+\infty$, if $x \notin F$) and by ψ_F^* its Fenchel conjugate, i.e. for all $p \in X^*$

$$\psi_F^*(p) = \sup_{x \in F} \langle p, x \rangle. \quad (1)$$

It is known that ψ_F^* is weak* lower semicontinuous convex continuous function. (The latter follows from the fact that the boundedness of F yields $\text{dom } \psi_F^* = X^*$). We also recall that every weak*-lower semicontinuous convex continuous function $g : X^* \rightarrow \mathbb{R} \cup \{+\infty\}$ coincides with the first conjugate f^* of a lower semicontinuous convex function f defined on X (take $f := g^*$). We denote by $\text{dom } g := \{p \in X^* : g(p) < +\infty\}$ the domain of the function g . Then, the Fenchel-Moreau subdifferential ∂g of g at any $p_0 \in \text{dom } g$ is defined as follows:

$$\partial g(p_0) = \{z^{**} \in X^{**} : g(p) - g(p_0) \geq \langle p - p_0, z^{**} \rangle, \forall p \in X^*\} \quad (2)$$

If $p_0 \in X^* \setminus \text{dom } g$, then we set $\partial g(p_0) = \emptyset$.

Given a closed subset F of X and a point x_0 of F we say that x_0 is a *strongly exposed* point of F , if there exists $p_0 \in X^*$ such that any sequence $\{x_n\}_{n \geq 1}$ in F satisfying $\lim_{n \rightarrow +\infty} \langle p_0, x_n \rangle = \sup_{x \in F} \langle p_0, x \rangle$ is converging to x_0 for the norm topology. In such case we say that the functional p_0 strongly exposes x_0 in F . We denote by $se(F)$ the set of strongly exposed points of F .

We now introduce the notion of a *weakly exposed* point, which will be useful in the sequel, see Lemma 5.

Definition 2 *Let F be a closed subset of X . A point $x_0 \in F$ is called weakly exposed point in F , if there exists $p_0 \in X^*$ such that any sequence $\{x_n\}_{n \geq 1}$ in F with $\lim_{n \rightarrow +\infty} \langle p_0, x_n \rangle = \sup_{x \in F} \langle p_0, x \rangle$ is weakly converging to x_0 .*

In the case of the above definition we say that the functional p_0 weakly exposes x_0 in F . It follows easily that p_0 attains its unique maximum on F at x_0 , hence in particular x_0 is an extreme point of F . We denote by $we(F)$ the set of weakly exposed points of F . Furthermore, a point x_0 is called a *point of continuity* of F , if the identity mapping $\text{id} : (F, \mathfrak{S}_w) \rightarrow (F, \mathfrak{S}_{\|\cdot\|})$ is continuous, where \mathfrak{S}_w (resp. $\mathfrak{S}_{\|\cdot\|}$) denotes the relative weak (resp. norm) topology of F . It follows directly that x_0 is a strongly exposed point of F if, and only if, it is both weakly exposed and a point of continuity of F . Finally, a point x_0 is called *weakly denting* (or *strongly extreme*, according to the terminology in [4, pg 67]), if for any relatively weakly open subset W in F containing x_0 there exist $p \in X^*$ and $\alpha > 0$ such that the set $\{x \in F : \langle p, x \rangle > \langle p, x_0 \rangle - \alpha\}$ is included in W .

3 Proof of the main result

The proof of Theorem 1 is based on the following result of Bourgain [3, Ch. 1; Th. 4]. (For a proof in English, see [4, Cor. 3.7.6].)

Theorem 3 *A Banach space X has the Radon-Nikodym property if, and only if, every nonempty closed convex bounded subset F of X has at least one weakly denting point.*

We can easily conclude the following corollary. The analogous result for dual Banach spaces is given in [8, Theorem 4].

Corollary 4 *For a Banach space X , the following are equivalent:*

- (i) X has the Radon-Nikodym property
- (ii) Every closed convex bounded subset of X is the closed convex hull of its weakly exposed points.
- (iii) Every nonempty closed convex bounded subset of X has at least one weakly exposed point.

Proof It is known ([4, Cor. 3.5.7], [10, Th. 5.21] e.g.) that a Banach space X has the Radon-Nikodym property if, and only if, every closed convex bounded subset of X is the closed convex hull of its strongly exposed points. This shows that (i) \implies (ii). Implication (ii) \implies (iii) is trivial, while (iii) \implies (i) follows from Theorem 3 and the observation that every weakly exposed point of F is weakly denting. \square

Remark 1 A weakly denting point is not in general weakly exposed, even in finite dimensions. Indeed, let $X = \mathbb{R}^2$, $F = \{(x_1, x_2) : f(x_1) \leq x_2 \leq g(x_1)\}$, where $f(x) = \max\{0, x^3\}$ and $g(x) = x + 1$, and $\bar{x} = (0, 0)$. Then \bar{x} is a weakly denting point of the compact convex set F , without being weakly exposed.

Remark 2 A Banach space X has the Radon-Nikodym property if, and only if, for every nonempty closed convex bounded subset F of X we have $\text{se}(F) \neq \emptyset$. However, if X does not have the Radon-Nikodym property, then the fact that $\text{we}(F) \neq \emptyset$ (or even that $\overline{\text{co}}(\text{we}(F)) = F$) for some closed convex bounded subset F of X does not necessarily imply that $\text{se}(F) \neq \emptyset$. (Consider the subset F of $c_0(\mathbb{N})$ defined by (12) in Proposition 8 and Claims 1 and 2 therein).

We shall finally need the following lemma.

Lemma 5 *Let X be a Banach space and F be any non-empty closed convex bounded subset X . Then the following are equivalent:*

- (i) The function ψ_F^* is Gâteaux differentiable at p_0 with derivative $x_0 \in X$.
- (ii) $x_0 \in F$ and the functional p_0 is weakly exposing x_0 in F .

Proof (i) \implies (ii) : Assume that (i) holds. Since $x_0 = \nabla^G \psi_F^*(p_0)$ (where $\nabla^G \psi_F^*$ denotes the Gâteaux derivative of ψ_F^*), we obviously have $x_0 \in \partial \psi_F^*(p_0)$, that is for all $p \in X^*$

$$\psi_F^*(p) - \psi_F^*(p_0) \geq \langle p - p_0, x_0 \rangle.$$

For $p = 0$ we obtain

$$\psi_F^*(p_0) := \sup_{x \in F} \langle p_0, x \rangle = \langle p_0, x_0 \rangle. \quad (3)$$

Let now $\{x_n\}_{n \geq 1}$ be a sequence in F such that

$$\lim_{n \rightarrow +\infty} \langle p_0, x_n \rangle = \sup_{x \in F} \langle p_0, x \rangle. \quad (4)$$

It suffices to show that $\{x_n\}_{n \geq 1}$ is weakly converging to x_0 . (Then, since the weak and the norm closure of the convex set F coincide, it will also follow that $x_0 \in F$).

Let us assume, towards a contradiction, that there exists a subsequence $\{x_{n_k}\}_{k \geq 1}$ of $\{x_n\}_{n \geq 1}$, $h \in X^*$ and $\alpha > 0$ such that for all $k \geq 1$

$$\langle h, x_{n_k} \rangle - \langle h, x_0 \rangle > \alpha. \quad (5)$$

Thanks to (3) and (4), we can consider $\varepsilon_n \searrow 0^+$ in a way that

$$\langle p_0, x_n \rangle \geq \langle p_0, x_0 \rangle - \varepsilon_n. \quad (6)$$

Since $\psi_F^*(p) \geq \langle p, x_n \rangle$, using (3) we get

$$\psi_F^*(p) \geq \psi_F^*(p_0) + \langle p, x_n \rangle - \langle p_0, x_0 \rangle,$$

which in view of (6) yields

$$\psi_F^*(p) \geq \psi_F^*(p_0) + \langle p - p_0, x_n \rangle - \varepsilon_n. \quad (7)$$

Set $t_n = 2\varepsilon_n/\alpha$. Then for $p = p_0 + t_n h$ relation (7) yields

$$(\psi_F^*)(p_0 + t_n h) - (\psi_F^*)(p_0) \geq \langle t_n h, x_n \rangle - \varepsilon_n$$

for all $n \geq 1$. In view of (5) this implies

$$\frac{(\psi_F^*)(p_0 + t_{n_k} h) - (\psi_F^*)(p_0)}{t_{n_k}} - \langle h, x_0 \rangle \geq \frac{\alpha}{2} > 0$$

for all $k \geq 1$. It follows that x_0 is not the Gâteaux derivative of ψ_F^* at p_0 , hence a contradiction.

(ii) \rightarrow (i) : Suppose that p_0 is weakly exposing x_0 in F , hence in particular $\langle p_0, x_0 \rangle = \sup_{x \in F} \langle p_0, x \rangle$. It follows easily from (2) that

$x_0 \in \partial\psi_F^*(p_0)$. Let us now suppose that (i) does not hold. Then there exist $\varepsilon > 0$, $h \in X^*$ with $\|h\| \leq 1$ and $t_n \searrow 0^+$ such that

$$(\psi_F^*)(p_0 + t_n h) - (\psi_F^*)(p_0) > \langle t_n h, x_0 \rangle + \varepsilon t_n. \quad (8)$$

For every $n \geq 1$, choose x_n in F such that

$$\langle p_0 + t_n h, x_n \rangle > (\psi_F^*)(p_0 + t_n h) - \frac{t_n}{n}. \quad (9)$$

Since $(\psi_F^*)(p_0) \geq \langle p_0, x_n \rangle$, the above inequality yields

$$\langle p_0 + t_n h, x_n \rangle - \langle p_0, x_n \rangle > (\psi_F^*)(p_0 + t_n h) - (\psi_F^*)(p_0) - \frac{t_n}{n}.$$

Hence

$$\langle t_n h, x_n \rangle > (\psi_F^*)(p_0 + t_n h) - (\psi_F^*)(p_0) - \frac{t_n}{n}. \quad (10)$$

Combining (8) and (10) we conclude

$$\langle h, x_n - x_0 \rangle > \varepsilon - \frac{1}{n},$$

which shows that $\{x_n\}_{n \geq 1}$ is not weakly converging to x_0 . However, since the sequence $\{x_n\}_{n \geq 1}$ is bounded and the function ψ_F^* is continuous, relation (9) yields $\lim_{n \rightarrow +\infty} \langle p_0, x_n \rangle = (\psi_F^*)(p_0)$, obtaining thus a contradiction to Definition 2. \square

Remark The above proof was inspired from techniques developed in [2] where a result between well-posed problems and differentiability was established. Results on the same spirit are also established in [6, Section 1], via a different approach. We are grateful to C. Zălinescu for bringing into our attention the aforementioned reference.

Proof of Theorem 1 The “only if” part follows from the result of Collier [5] and the fact that the Fréchet derivatives of weak*-lower semicontinuous convex continuous on X^* always belong to the predual space X (see [7] e.g.).

For the “if” part, let F be any closed convex bounded subset of X . Then the function ψ_F^* of X^* (given in (1)) is weak*-lower semicontinuous convex and continuous. From our hypothesis and Lemma 5 we conclude that $\text{we}(F) \neq \emptyset$. Since F is arbitrary, Corollary 4 asserts that X has the Radon-Nikodym property. \square

Let us recall that a Banach space X is called *weakly sequentially complete*, if every weakly Cauchy sequence of X is weakly converging in X . A typical example of a non-reflexive weakly sequentially complete Banach space is the space $L^1(\mu)$, where μ is a σ -finite positive measure. The following remark is due to G. Godefroy.

Corollary 6 *Let X be a weakly sequentially complete Banach space. Then X has the Radon-Nikodym property if, and only if, every weak*-lower semicontinuous convex continuous function on X^* is Gâteaux differentiable at some point of its domain.*

Proof The “only if” part is a direct consequence of Theorem 1. The “if” part follows from the following observation: if F is a nonempty closed convex bounded subset of X , and if $\nabla^G \psi_F^*(p)$ is the Gâteaux derivative of the function ψ_F^* at $p \in X^*$, then there exists $\{x_n\}_{n \geq 1}$ in F that is weakly*-converging to $\nabla^G \psi_F^*(p)$ (see proof of Lemma 5 (i) \implies (ii)). It follows that $\{x_n\}_{n \geq 1}$ is a weakly Cauchy sequence, hence in view of our hypothesis $\nabla^G \psi_F^*(p) \in X$. (For similar considerations, see also [9].) We conclude by Lemma 5 (i) \implies (ii) and Corollary 4 (iii) \implies (i). \square

Lemma 5 has also the following consequence. (The proof below is similar to [10, Theorem 5.20]).

Corollary 7 *Let F be a closed convex bounded subset of X . If ψ_F^* is Gâteaux differentiable in a dense subset of X^* with derivatives in X , then $F = \overline{\text{co}}(\text{we}(F))$.*

Proof Since F is bounded, we have $\text{dom}(\psi_F^*) = X^*$. (In particular the function ψ_F^* is convex and Lipschitz). Since F is closed and convex, we have $\overline{\text{co}}(\text{we}(F)) \subseteq F$. Let us suppose, towards a contradiction, that there exists some x_0 in $F \setminus \overline{\text{co}}(\text{we}(F))$. Then applying the Hahn-Banach theorem, we can find $p \in X^*$ ($p \neq 0$) and $\alpha \in \mathbb{R}$ such that

$$\sup \{ \langle p, x \rangle : x \in \overline{\text{co}}(\text{we}(F)) \} < \alpha < \langle p, x_0 \rangle.$$

Set $D = \{q \in X^* : \exists \nabla^G(\psi_F^*)(q) \in X\}$. Since D is dense in X^* , we may find $q \in D$ close to p such that $z := \nabla^G(\psi_F^*)(q) \in X$ and

$$\sup \{ \langle q, x \rangle : x \in \overline{\text{co}}(\text{we}(F)) \} < \alpha < \langle q, x_0 \rangle. \quad (11)$$

By Lemma 5 we conclude that $z \in \text{we}(F)$ and that the functional q weakly exposes z . This clearly contradicts (11). \square

The space $c_0(\mathbb{N})$ is a typical example of a Banach space without the Radon-Nikodym property. In this case, as already mentioned in Section 1, the norm $\|\cdot\|_1$ provides an example of a weak*-lower semicontinuous convex continuous function of $\ell^1(\mathbb{N})$, which is generically Gâteaux differentiable with all derivatives in $X^{**} \setminus X$. In the following proposition we give an example of a (nowhere Fréchet differentiable) weak*-lower semicontinuous convex continuous function of $\ell^1(\mathbb{N})$, which is Gâteaux differentiable with derivatives in the predual space in a dense set.

Proposition 8 *Let $X = c_0(\mathbb{N})$. Then there exists a weak*-lower semicontinuous convex continuous function $f : X^* \rightarrow \mathbb{R}$ such that:*
(i) there exists a dense subset D of X^ such that f is Gâteaux differentiable at every point of D with derivative in the predual space ;*
(ii) f is nowhere Fréchet differentiable.

Proof Set $X = c_0(\mathbb{N})$ and consider the set

$$F = B_X \cap c_0^+(\mathbb{N}) := \{x = (x^i)_i : \|x\|_\infty \leq 1 \text{ and } x^i \geq 0 (\forall i \in \mathbb{N})\}. \quad (12)$$

It is easily seen that F is closed convex bounded and that

$$\text{ext}(F) = \{x \in F : x^i \in \{0, 1\} \text{ for all } i\}$$

where $\text{ext}(F)$ denotes the set of the extreme points of F .

CLAIM 1 *Let $\bar{x} \in \text{ext}(F)$ and consider the finite set*

$$I_{\bar{x}} = \{i \in \mathbb{N} : \bar{x}^i = 1\}. \quad (13)$$

Then any functional $p = (p^i)_i$ of $X^ := \ell^1(\mathbb{N})$ satisfying*

$$\begin{aligned} p^i &> 0, & \text{if } i \in I_{\bar{x}} \\ p^i &< 0, & \text{if } i \in \mathbb{N} \setminus I_{\bar{x}} \end{aligned} \quad (14)$$

weakly exposes the point \bar{x} .

In particular $\text{ext}(F) = \text{we}(F)$ (hence $\text{we}(F) \neq \emptyset$).

[Proof of Claim 1: Let $\bar{x} \in \text{ext}(F)$, $I_{\bar{x}} = \{i \in \mathbb{N} : \bar{x}^i = 1\}$ and consider any p in $\ell^1(\mathbb{N})$ satisfying (14). We first note that for all $x \in F$ and all $i \in \mathbb{N}$ we have

$$p^i x^i \leq p^i \bar{x}^i. \quad (15)$$

It follows that $\langle p, x \rangle \leq \langle p, \bar{x} \rangle$, for all $x \in F$, i.e. $\langle p, \bar{x} \rangle = \sup_{x \in F} \langle p, x \rangle$.
 Take now any sequence $\{x_n\}_{n \geq 1}$ in F such that $\lim_{n \rightarrow +\infty} \langle p, x_n \rangle = \langle p, \bar{x} \rangle$.

Let us show that

$$\lim_{n \rightarrow +\infty} x_n^i = \bar{x}^i \quad (16)$$

for all $i \geq 0$.

Indeed, assume that for some i_0 (16) does not hold. Then there exist a subsequence $\{x_{n_k}^{i_0}\}_{k \geq 1}$ of $\{x_n^{i_0}\}_{n \geq 1}$, $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$|x_{n_k}^{i_0} - \bar{x}^{i_0}| > \frac{\varepsilon}{|p^{i_0}|}.$$

Using (15) we infer that

$$p^{i_0} x_{n_k}^{i_0} < p^{i_0} \bar{x}^{i_0} - \varepsilon.$$

Combining with (15) we get $\langle p, x_{n_k} \rangle < \langle p, \bar{x} \rangle - \varepsilon$, for all $k \geq k_0$. This contradicts the fact that $\langle p, x_n \rangle \rightarrow \langle p, \bar{x} \rangle$. It follows that (16) holds for all $i \geq 0$.

Since the sequence $\{x_n\}_{n \geq 1}$ is bounded, we conclude from (16) that $x_n \xrightarrow{w} \bar{x}$. Hence the functional p is weakly exposing \bar{x} in F . Since every weakly exposed point is obviously extreme, the proof of the claim is complete. \blacklozenge]

CLAIM 2 $\text{se}(F) = \emptyset$.

[Proof of Claim 2: It clearly suffices to show that any point \bar{x} in $\text{we}(F)$ is not a point of continuity for F . To this end, take any $\bar{x} \in \text{we}(F)$ and consider the sequence $\{x_n\}_{n \geq 1}$ in F with

$$x_n^i = \begin{cases} 1 & \text{if } i \in I_{\bar{x}} \cup \{n\} \\ 0 & \text{elsewhere} \end{cases}$$

where $I_{\bar{x}}$ is given by (13). Then it follows easily that $x_n \xrightarrow{w} \bar{x}$. On the other hand, for n sufficiently large, we have $\|x_n - \bar{x}\|_\infty = 1$. \blacklozenge]

Consider now the weak*-lower semicontinuous convex continuous function $\psi_F^* : \ell^1(\mathbb{N}) \rightarrow \mathbb{R}$ defined by

$$\psi_F^*(p) := \sup_{x \in F} \langle p, x \rangle = \|p_+\|_1 \quad (17)$$

where $\|\cdot\|_1$ is the usual norm of $\ell^1(\mathbb{N})$ and

$$p_+^i = \begin{cases} p^i & \text{if } p^i > 0 \\ 0 & \text{if } p^i \leq 0. \end{cases}$$

Let us denote by D the set of all functionals $p = (p^i)_i$ in $\ell^1(\mathbb{N})$ satisfying (14) for some finite (possibly empty) subset I of \mathbb{N} . For every such functional p , consider the point $\bar{x} = (\bar{x}^i)_i$ of $c_0(\mathbb{N})$ defined by

$$\bar{x}^i = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{if } i \in \mathbb{N} \setminus I. \end{cases}$$

Then $\bar{x} \in F$ and $I = I_{\bar{x}}$ (where $I_{\bar{x}}$ is given in (13)). It follows by Claim 1 that the functional p weakly exposes \bar{x} . Applying Lemma 5 (ii) \rightarrow (i) we conclude that \bar{x} is the Gâteaux derivative of ψ_F^* at p .

Let us now show that D is dense in $\ell^1(\mathbb{N})$. Indeed, let any $q = (q^i)_i$ in $\ell^1(\mathbb{N})$ and any $\varepsilon > 0$. Then for some $n_0 \in \mathbb{N}$ we have:

$$\|q\|_1 \leq \sum_{i=0}^{n_0} |q^i| + \frac{\varepsilon}{2}.$$

Consider now the functional $p = (p^i)_i$ defined by

$$p^i = \begin{cases} q^i & \text{if } i \leq n_0 \text{ and } q^i \neq 0 \\ -\frac{\varepsilon}{2^{i+2}} & \text{elsewhere.} \end{cases}$$

It is easily seen that $p \in D$. Moreover,

$$\|q - p\|_1 = \sum_{i=0}^{+\infty} |q^i - p^i| \leq \sum_{i>n_0} |q^i| + \sum_{i=0}^{+\infty} \frac{\varepsilon}{2^{i+2}} \leq \varepsilon.$$

We have shown that the function ψ_F^* is densely Gâteaux differentiable with derivatives in the predual space X . On the other hand, since by Claim 2 the set F has no strongly exposed points, it follows from [1, p. 450] that ψ_F^* is nowhere Fréchet differentiable.

Let us finally note that the function ψ_F^* is in fact generically Gâteaux differentiable. Indeed, it is easily seen that for every $p = (p^i)_i$ with $p^i \neq 0$ for all i , we have $\nabla^G \psi_F^*(p) = z^{**}$ where $z^{**} \in \ell^\infty(\mathbb{N})$ is given by

$$(z^{**})^i = \begin{cases} 1 & \text{if } p^i > 0 \\ 0 & \text{if } p^i < 0 \end{cases}$$

□

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References

- [1] ASPLUND, E. & ROCKAFELLAR R.T., Gradients of convex functions, *Trans. Amer. Math. Soc.* **139** (1969), 433-467.
- [2] BACHIR, M., On generic differentiability and Banach-Stone's theorem, preprint 37p, University of Bordeaux, 1999 (in abbreviate version: *C. R. Acad. Sci. Paris* **330** (2000), 687-690.)
- [3] BOURGAIN, J., *La propriété de Radon-Nikodym*, Publ. Math. de l'Univ. Pierre et Marie Curie **36**, 1979.
- [4] BOURGIN, R., *Geometric Aspects of Convex Sets with the Radon-Nikodym Property*, Lecture Notes in Mathematics **993** (Springer Verlag, Berlin 1983).
- [5] COLLIER, J., The dual of a space with the Radon-Nikodym property, *Pacific J. Math.* **64** (1976), 103-106.
- [6] DONTCHEV, A. & ZOLEZZI, T., *Well-Posed Optimization Problems*, Lecture Notes in Mathematics **1543** (Springer Verlag, Berlin 1993).
- [7] FABIAN, M. & ZIZLER, V., An elementary approach to some questions in higher order smoothness in Banach spaces, *Extracta mathematicae* (to appear).
- [8] GILES, J., Comparable differentiability characterizations of two classes of Banach spaces, *Bull. Austral. Math. Soc.* **56** (1997), 263-272.
- [9] GODEFROY, G., Propriété de lissité de certaines normes, *Math. Annalen* **257** (1981), 185-189.
- [10] PHELPS, R., *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Mathematics **1364** (2nd ed.) (Springer Verlag, Berlin 1993).

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