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GRADIENT FLOWS, SECOND-ORDER GRADIENT SYSTEMS AND CONVEXITY*

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Abstract. We disclose an interesting connection between the gradient flow of a $\mathscr{C}^2$-smooth function ψ and strongly evanescent orbits of the second-order gradient system defined by the square-norm of $\nabla\psi$, under an adequate convexity assumption. As a consequence, we obtain the following surprising result for two $\mathscr{C}^2$, convex, and bounded from below functions ψ_1, ψ_2: if $||\nabla\psi_1|| = ||\nabla\psi_2||$, then $\psi_1 = \psi_2 + k$ for some $k \in \mathbb{R}$.

Key words. gradient flows, second-order gradient equations, convexity, dynamical systems

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1. Introduction.

We are interested in the first-order gradient system

$$u'(t) = -\nabla\psi(u(t)), \quad t \geq 0, \tag{DS-1}$$

in comparison with the second-order gradient system

$$v''(t) = \nabla V(v(t)), \quad t \geq 0, \tag{DS-2}$$

where $\psi : \mathcal{H} \to \mathbb{R}$ is a $\mathscr{C}^2$ function, $V : \mathcal{H} \to \mathbb{R}$ is a $\mathscr{C}^1$ function, $\nabla\psi$, ∇V denote the respective gradients and $\mathcal{H}$ stands for a Hilbert space with inner product $\langle \cdot \mid \cdot \rangle$ and associated norm $\|\cdot\|$. Throughout this work, the functions ψ and V will be linked by the relation

$$V(x) = \frac{1}{2}\,\|\nabla\psi(x)\|^2, \quad x \in \mathcal{H}. \tag{1}$$

The second-order system (DS-2) is introduced here (and studied for the potential V given by (1)) for the first time in the literature.

In what follows the set of critical points of ψ (singular set) will be denoted by

$$\mathrm{Crit}_\psi = \{x \in \mathcal{H} \mid \nabla\psi(x) = 0\} = \{x \in \mathcal{H} \mid V(x) = 0\}.$$

When ψ is convex, the set Crit_ψ is convex and consists of all (global) minimizers of ψ. Therefore, in this case the set of critical values $\psi(\mathrm{Crit}_\psi)$ is either empty or a singleton. We may also observe that Crit_ψ is also the set of minimizers of V. Therefore, it is also convex whenever V is assumed so.

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By a *global* solution of (DS-1) (respectively, (DS-2)) we mean a function $u \in \mathscr{C}^1([0,+\infty), H)$ (respectively, $v \in \mathscr{C}^2([0,+\infty), H)$) satisfying (DS-1) (respectively, (DS-2)) for all $t \geq 0$. In both cases, we impose the initial condition

$$(\mathrm{I}_0) \qquad u(0) = u_0 \qquad (\text{respectively, } v(0) = u_0)$$

for some given $u_0 \in \mathcal{H}$. This is a very common way to obtain unique solutions for (DS-1), whereas for (DS-2) an additional condition on the initial velocity $v'(0)$ is normally required. We deliberately refrain from doing so, and instead require the solutions of (DS-2) to be global on $[0,+\infty)$ and to comply with one of the following *asymptotic conditions*, introduced in the following definition.

Definition 1.1 (weakly and strongly evanescent solutions). *A global solution v of* (DS-2) *is called*

- *weakly evanescent (in short, w-evanescent) if it satisfies*

$$(\text{w-EV}) \qquad \liminf_{t\to+\infty} \|v'(t)\| = \liminf_{t\to+\infty} V(v(t)) = 0,$$

- *strongly evanescent if it satisfies*

$$\|v'(\cdot)\| \in L^2(0,+\infty) \quad \textit{and} \quad V(v(\cdot)) \in L^1(0,+\infty),$$

or equivalently

$$(\text{EV}) \qquad \int_0^{+\infty} \left(\|v'(t)\|^2 + V(v(t)) \right) dt \, < \, +\infty.$$

Remark 1.2. (i) Conditions (w-EV) and (EV) as well as the associated terminology appear to be new in the literature. Both conditions correspond to a kind of boundary condition of the orbit $v(t)$ at infinity. (ii) Any strongly evanescent solution of (DS-2) is also w-evanescent.

It is straightforward to see that any global solution of (DS-1) is also a solution of (DS-2). However, this solution might fail to satisfy (EV). To see this, let $n = 1$ and $\psi(x) = -x^2$, for $x \in \mathbb{R}$, and notice that $v(t) = e^{2t}x_0$ is a solution of (DS-1) (and consequently of (DS-2)), but (EV) fails, since $v \notin L^2(0,+\infty)$. Conversely, a solution of (DS-2) satisfying (EV) and (I_0) might not be a solution of (DS-1) since the system (DS-2)–(EV) does not distinguish between ψ and $-\psi$.

Let us further consider the following two conditions:

$$(C) \ \inf_{z\in\mathcal{H}} \|\nabla\psi(z)\| = 0 \qquad \text{and} \qquad (C^\star) \ \ \psi \text{ is bounded below.}$$

By Ekeland's variational principle [20, Corollary 2.3] we deduce that $(C^\star) \Longrightarrow (C)$. This latter condition (C) is necessary for the existence of w-evanescent solutions of (DS-2).

A constant function $v = \hat{x}$ is a w-evanescent solution of (DS-2) if and only if $\hat{x} \in \mathrm{Crit}_\psi$, while $\mathrm{Crit}_\psi \neq \emptyset$ clearly implies (C). If in addition ψ is convex, then $(C^\star)$ is also fulfilled. The example of the convex $\mathscr{C}^2$ function

$$(2) \qquad \psi(x) = \begin{cases} -\ln(1-x) & \text{if } x \leq 0, \\ \frac{1}{2}x^2 + x & \text{if } x \geq 0 \end{cases}$$

shows that (C) and $(C^\star)$ are not equivalent, despite the fact that ψ is convex (in this case, only (C) holds).

Description of the results. First-order and second-order gradient systems have often been explored independently in the literature (see, e.g., [12, 14, 23, 25, 22, 15, 18, 24, 3] and references therein). A first innovative aspect of this work is to introduce the particular second-order ordinary differential equation (DS-2), for a potential $V(\cdot)$ given by (1), and shed light on its connection with the first-order gradient system (DS-1) when either f or V is convex. Exploring this link reveals some unexpected properties of convex functions described below. Another by-product, as we shall see, concerns uniqueness of smooth solutions to a certain eikonal equation.

More precisely, in this work we show that if either ψ or V is *convex*, then any solution of (DS-2) satisfying (I_0)–(EV) is also a solution of (DS-1)–(I_0), and vice versa. In particular, the second-order system (DS-2) coupled with (I_0)–(EV) is well posed and can be integrated to obtain the first-order system (DS-1). An important consequence of this result is an intimate link between convexity properties of ψ and of $\|\nabla\psi\|^2$ (Corollary 3.17):

$$(\|\nabla\psi\|^2 \text{ convex and } \psi \text{ bounded below}) \Longrightarrow \psi \text{ convex.}$$

This leads to the surprising corollary

$$\|\nabla\psi_1\| = \|\nabla\psi_2\| \Longrightarrow \psi_1 = \psi_2 + \text{constant},$$

provided that one of the following assumptions is fulfilled:

(a) ψ_1 and ψ_2 are convex and $\inf \|\nabla\psi_1\| = 0$ (Theorem 3.8),

(b) $\|\nabla\psi_1\|^2$ is convex and ψ_1 and ψ_2 are bounded below (Corollary 3.20).

Another consequence is a uniqueness property for smooth solutions of the usual eikonal equation

$$\|\nabla\psi\|^2 = f \tag{3}$$

in *the whole space*. It is well known that uniqueness plays a prominent role in understanding the structure of the set of solutions of (3) (see, e.g., [27, 30, 16, 17, 29, 8, 32, 9, 21, 26, 10] and references therein). Here we obtain uniqueness of bounded below $\mathscr{C}^2$ solutions when f is nonnegative and convex. When f is only nonnegative, we prove that (3) has at most one bounded from below $\mathscr{C}^2$ convex solution. If f is only nonnegative, we prove that any convex and bounded below solution is unique.

Finally, disclosing the link between (DS-1) and (DS-2) leads to a simple variational principle for the first-order gradient system (DS-1) when $\|\nabla\psi\|^2$ is convex and ψ is bounded below (Proposition 3.22).

Structure of the manuscript. The rest of the paper is organized as follows. In section 2 we recall basic properties of the first-order system (DS-1) for $\psi \in \mathscr{C}^2(\mathcal{H})$ and for the second-order system (DS-2) for $V(x) = \frac{1}{2}||\nabla\psi(x)||^2$ that will be used in what follows. No originality is claimed in subsection 2.1 or in the beginning of subsection 3.1, where most of the stated properties of the first-order system (DS-1) are essentially known. These properties are recalled for completeness, and short proofs are eventually provided to keep the manuscript self-contained. Subsection 2.2 contains properties of the system (DS-2) with emphasis on Lyapunov functions and on asymptotic behavior of the orbits, while subsection 2.3 is dedicated to comparing the solutions of these two systems.

The main results are given in section 3 and organized as follows: subsection 3.1 collects all results obtained under the driving assumption that ψ is convex, while subsection 3.2 does the same under the assumption that V is convex. We quote in particular Theorem 3.8 (determination of a convex function by the modulus of its gradient) and its variant Corollary 3.20, which are important consequences of Theorem 3.6 (equivalence of solutions of (DS-1) and (DS-2) if ψ is convex) and Proposition 3.16, respectively. Finally, in subsection 3.3 we associate with the first-order system (DS-1) an alternative variational principle, which is in the spirit of the results of this work.

We assume familiarity with basic properties and characterizations of convex functions. These prerequisites can be found in the classical books [35] or [36].

2. Basic properties of first- and second-order gradient systems.

2.1. First-order gradient systems: Basic properties. In this subsection we recall for completeness basic properties of solutions of the first-order gradient system (DS-1), which will be used in what follows. In this subsection the functions $\psi \in \mathscr{C}^2(\mathcal{H})$ and $V(\cdot)$ given in (1) are not yet assumed to be convex.

LEMMA 2.1 (Lyapunov for (DS-1)). *Let $u(\cdot)$ be a maximal solution of* (DS-1) *defined on $[0, T_{\max})$, where $T_{\max} \in (0, +\infty]$. Then,*

(i) *$\rho(t) := \psi(u(t))$ is nonincreasing on $[0, T)$ and, for every $T < T_{\max}$,*

$$\int_0^T \|u'(t)\|^2 \, dt = \rho(0) - \rho(T); \tag{4}$$

(ii) *$\|u'(\cdot)\| \in L^2(0, T_{\max})$ if and only if*

$$\inf_{0 \le t < T_{\max}} \psi(u(t)) > -\infty. \tag{5}$$

Proof. Since $\rho'(t) = \langle \nabla\psi(u(t)) \mid u'(t)\rangle = -\|u'(t)\|^2 = -\|\nabla\psi(u(t))\|^2 \le 0$, we deduce (i). The second assertion follows by taking the limit as $T \to T_{\max}$. □

Remark 2.2 (strict Lyapunov). Assuming $\psi \in \mathscr{C}^2(\mathcal{H})$ yields that both (DS-1) and the equation $w'(t) = \nabla\psi(w(t))$ admit unique solutions under a given initial condition. A standard argument now shows that if the initial condition is not a singular point (that is, $\nabla\psi(u(0)) \neq 0$), then $\nabla\psi(u(t)) \neq 0$ for every $t > 0$ and ρ is strictly decreasing.

LEMMA 2.3 (maximal nonglobal solutions). *If $u(\cdot)$ is a maximal solution of* (DS-1) *which is not global (i.e., $T_{\max} < +\infty$), then*

$$\inf_{0 \le t < T_{\max}} \psi(u(t)) = \lim_{t \to T_{\max}} \psi(u(t)) = -\infty \tag{6}$$

and

$$\int_0^{T_{\max}} \|u'(t)\|^2 \, dt = +\infty. \tag{7}$$

Proof. In view of Lemma 2.1(i), assertions (6) and (7) are equivalent. Assume now that (7) does not hold. Then the integral

$$\int_0^{T_{\max}} u'(t) dt,$$

converges in $\mathcal{H}$ to the element $u(T_{\max}) - u_0$, where $u(T_{\max}) = \lim_{t \to T_{\max}} u(t)$. Moreover $\nabla\psi(u(T_{\max})) \neq 0$ (cf. Remark 2.2). Considering the Cauchy problem $w'(t) = -\nabla\psi(w(t))$ with initial condition $w(T_{\max}) = u(T_{\max})$, we deduce that the (presumably maximal) solution $u(\cdot)$ can be extended to the right on an interval of the form $[0, T_{\max} + \varepsilon)$ for some $\varepsilon > 0$, which is a contradiction. □

COROLLARY 2.4. *If ψ is bounded below, then any maximal solution $u(\cdot)$ of* (DS-1) *is global and $\|u'(\cdot)\| \in L^2(0,+\infty)$.*

Proof. If ψ is bounded below, then (6) cannot be satisfied and the solution u is global. Obviously, (5) is fulfilled, yielding $\|u'(\cdot)\| \in L^2(0,+\infty)$. ∎

Remark 2.5 (grad-coercive functions). A function $\psi \in \mathscr{C}^1(\mathcal{H})$ is called *grad-coercive* if $\|\nabla\psi\|$ is bounded on the sublevel sets $[\psi \le \alpha] := \{x \in \mathcal{H} : \psi(x) \le \alpha\}$, $\alpha \in \psi(\mathcal{H})$.

If ψ is grad-coercive, then any maximal solution of (DS-1) is global. Indeed, let $u(\cdot)$ be a maximal solution defined on $[0, T_{\max})$. Since $u(t) \in [\psi \le \psi(u(0))]$, for all $t \in [0, T_{\max})$, the function $\|\nabla\psi(u(\cdot))\|$ is bounded on $[0, T_{\max}]$. Setting $M = \sup_{0\le t<T_{\max}} \|\nabla\psi(u(t))\|$, we obtain

$$\int_0^{T_{\max}} \|u'(t)\|^2\,dt = \int_0^{T_{\max}} \|\nabla\psi(u(t))\|^2\,dt \le MT_{\max} < +\infty,$$

which contradicts (7). ∎

Let us observe that ψ can be grad-coercive without being bounded from below. A simple example is the identity function $x \mapsto x$ on $\mathbb{R}$. Similarly, a function which is bounded below is not necessarily grad-coercive, for example the function $x \mapsto \cos(x^2)$.

Remark 2.6 (relation to other domains). The asymptotic behavior of (DS-1) has been studied by several authors in the framework of analytic geometry (see, e.g., [28, 37]), in relation to convexity (see [7, 18, 19, 31]), to optimization algorithms (see, e.g., [2, 4, 11]) and to PDEs (see, e.g., [15, 24]). Roughly speaking, good asymptotic behavior requires a strong structural assumption (analyticity or convexity); see [1] or [34, Page 12] for classical counterexamples.

2.2. Second-order systems: Properties of strongly evanescent solutions. In this subsection we emphasize properties of weakly and strongly evanescent solutions of the second-order system (DS-2), where $\psi \in \mathscr{C}^2(\mathcal{H})$ and

$$V(x) = \frac{1}{2}||\nabla\psi(x)||^2.$$

LEMMA 2.7 (equality of modula). *Let $v(\cdot)$ be a w-evanescent solution of* (DS-2). *Then*

$$\|v'(t)\| = \|\nabla\psi(v(t))\| \quad \text{for all } t \ge 0. \tag{8}$$

Proof. It is easily seen that $I(t) := \frac{1}{2}\|v'(t)\|^2 - V(v(t))$ is a first integral of the system (DS-2), that is, for some $k \in \mathbb{R}$ and all $t \ge 0$ it holds that $\|v'(t)\|^2 = k + 2V(v(t))$. Taking the limit inferior as $t \to +\infty$ we infer from (w-EV) that $k = 0$ and the result follows. ∎

LEMMA 2.8 (range of orbits). *If* $\mathrm{Crit}_\psi = \emptyset$*, then the range $\{v(t)\,;\, t \ge 0\}$ of any w-evanescent solution $v(\cdot)$ of* (DS-2) *cannot be relatively compact.*

Proof. Let $v(\cdot)$ be a w-evanescent solution of (DS-2). If $\{v(t)\,;\, t \ge 0\}$ were relatively compact, then there would exist a sequence $(t_n)_{n\ge 0}$ such $v(t_n) \to z_0$ for some $z_0 \in \mathcal{H}$. By (w-EV) we obtain $V(z_0) = 0$. Therefore $\nabla\psi(z_0) = 0$, that is, $\mathrm{Crit}_\psi \ne \emptyset$, a contradiction. ∎

The following proposition assembles properties of the strongly evanescent solutions of (DS-2).

Proposition 2.9 (properties of strongly evanescent solutions). *Let $v(\cdot)$ be a strongly evanescent solution of* (DS-2)*. Then the following hold.*

(i) *We have* $\lim_{t\to+\infty} \psi(v(t)) \in \mathbb{R}$ *and*

$$|\psi(v(0)) - \psi(v(t))| \leq \int_0^t \|v'(s)\|^2 \, ds \quad \text{for all } t \geq 0. \tag{9}$$

(ii) *If ψ is coercive (i.e., $[\psi \leq \alpha]$ is bounded for all $\alpha \in \psi(\mathcal{H})$), then $v(\cdot)$ is bounded.*

(iii) *If $\left\|\nabla^2\psi(v(\cdot))\right\|$ is bounded, then* $\lim_{t\to+\infty} \|v'(t)\| = \lim_{t\to+\infty} V(v(t)) = 0$.

(iv) *The function*

$$\phi(t) := v'(t) + \sigma\nabla\psi(v(t)), \quad \sigma \in \{-1, 1\},$$

satisfies

$$\phi'(t) = \sigma\nabla^2\psi(v(t))\phi(t).$$

Proof. Set $r(t) := \psi(v(t))$, $t \geq 0$. Then $|r'(t)| = \langle v'(t) \mid \nabla\psi(v(t))\rangle$. By the Cauchy–Schwarz inequality and Lemma 2.7 we get

$$|r'(t)| \leq \|v'(t)\| \, \|\nabla\psi(v(t))\| \; = \; \|v'(t)\|^2.$$

Thus, in view of (EV), $r' \in L^1(0, +\infty)$ and the limit $\lim_{t\to+\infty} r(t) = \lim_{t\to+\infty} \psi(v(t))$ exists. Moreover, we have

$$|r(t) - r(0)| \leq \int_0^t \|v'(s)\|^2 \, ds \; \leq \; \int_0^{+\infty} \|v'(s)\|^2 \, ds \; < \; +\infty.$$

We easily deduce that the range $\{r(t) : t \geq 0\}$ is bounded, yielding $v(t) \in [\psi \leq \eta]$ for some $\eta > 0$ and all $t \geq 0$. Therefore (ii) holds. Differentiating the function $V(x) = \frac{1}{2}||\nabla\psi(x)||^2$ and substituting $x = v(t)$ we deduce

$$\|\nabla V(v(t))\| \leq \left\|\nabla^2\psi(v(t))\right\| \|\nabla\psi(v(t))\| \, . \tag{10}$$

On the other hand,

$$\left|\frac{d}{dt}\left[V(v(t))\right]\right| = |\langle\nabla V(v(t)) \mid v'(t)\rangle| \; \leq \; \|\nabla V(v(t))\| \, \|v'(t)\|. \tag{11}$$

Combining (10) with (11) and recalling (8) and the definition of V we get

$$\left|\frac{d}{dt}\left[V(v(t))\right]\right| \; \leq \; 2 \, \left\|\nabla^2\psi(v(t))\right\| \, V(v(t)). \tag{12}$$

Since $v(\cdot)$ is strongly evanescent, $V(v(\cdot)) \in L^1(0, +\infty)$, while $\left\|\nabla^2\psi(v(\cdot))\right\|$ is bounded by assumption. We deduce from (12) that $\frac{d}{dt}[V(v(\cdot))] \in L^1(0, +\infty)$. Therefore the limit $\lim_{t\to+\infty} V(v(t))$ exists (and necessarily equals zero, since $V(v(\cdot)) \in L^1(0, +\infty)$). Thus (iii) holds. Finally, (iv) follows from direct calculation, using (DS-2) and (1). ∎

The following proposition will be used in what follows.

Proposition 2.10 (further asymptotic properties of strongly evanescent solutions). *Let $v(\cdot)$ be a strongly evanescent solution of* (DS-2)*, where V is given by* (1)*. Then*

$$\frac{||v(t) - v(0)||}{t}, \; \frac{\|v(t)\|}{\sqrt{t^2 + 1}} \in L^2(0, +\infty), \quad \lim_{t\to+\infty} \frac{\|v(t)\|}{\sqrt{t}} = 0, \tag{13}$$

and for every $t \geq 0$ *it holds that*

$$\int_0^t \frac{\|v(t)-v(0)\|^2}{t^2}\,dt \;\leq\; 4\int_0^t \|v'(t)\|^2\,dt. \tag{14}$$

Proof. (i) Set $w(t) = v(t) - v(0)$, $t \geq 0$ (therefore $\lim_{t\to 0^+} \frac{w(t)}{t} = v'(0)$). Integrating by parts and using the Cauchy–Schwarz inequality we obtain, for every $t > 0$,

$$\int_0^t \frac{\|w(s)\|^2}{s^2}\,ds = -\frac{\|w(t)\|^2}{t} + 2\int_0^t \frac{\langle w(s) \mid w'(s)\rangle}{s}\,ds \;\leq\; 2\int_0^t \frac{\langle w(s) \mid w'(s)\rangle}{s}\,ds$$
$$\leq 2\left(\int_0^t \frac{\|w(s)\|^2}{s^2}\,ds\right)^{1/2}\left(\int_0^t \|w'(s)\|^2\,ds\right)^{1/2},$$

yielding

$$\int_0^t \frac{\|w(s)\|^2}{s^2}\,ds \leq 4\int_0^t \|w'(s)\|^2\,ds = 4\int_0^t \|v'(s)\|^2\,ds.$$

Therefore, (14) follows. In particular, since $v(\cdot)$ is a strongly evanescent solution, we conclude that $t^{-1}\|w(t)\| \in L^2(0,+\infty)$ (hence, $(t^2+1)^{-1/2}\|w(t)\| \in L^2(0,+\infty)$). Since $(t^2+1)^{-1/2} \in L^2(0,+\infty)$, we deduce easily that $(t^2+1)^{-1/2}\|v(t)\| \in L^2(0,+\infty)$.

(ii) Fix $t_0 > 0$. Then for all $t > t_0$ we have

$$\int_{t_0}^t \frac{\|v(s)\|^2}{s^2}\,ds = -\frac{\|v(t)\|^2}{t} + \frac{\|v(t_0)\|^2}{t_0} + 2\int_{t_0}^t \frac{\langle v(s) \mid v'(s)\rangle}{s}\,ds.$$

Both integrals in the above expression converge as $t \to +\infty$, yielding that the limit $\lim_{t\to+\infty} \frac{\|v(t)\|^2}{t}$ also exists. This limit is zero since $t^{-1}\|v(t)\| \in L^2(t_0,+\infty)$. □

2.3. Comparison of solutions of (DS-1) and (DS-2). We now focus our attention upon comparison of solutions of the first-order system (DS-1) and evanescent solutions of the second-order gradient system (DS-2), where $\psi \in \mathscr{C}^2(\mathcal{H})$ and V is given by (1).

The following result states that each solution $u(\cdot)$ of (DS-1) is also a strongly evanescent solution of (DS-2) unless $\lim_{t\to+\infty} \psi(u(t)) = -\infty$. As underlined in the introduction, the inverse is more complicated: in general, strongly evanescent solutions of (DS-2) are not necessarily solutions of (DS-1). Surprisingly, under a convexity assumption on either ψ or V, strongly evanescent solutions of (DS-2) are also solutions of (DS-1).

LEMMA 2.11 (characterization of w-evanescent/strongly evanescent solutions). *Let* $u(\cdot)$ *be a global solution of* (DS-1). *Then,*

(i) *u is a global solution of* (DS-2),

(ii) *u is a w-evanescent solution of* (DS-2) *if and only if*

$$\inf_{t\geq 0} \|\nabla\psi(u(t))\| = \inf_{z\in\mathcal{H}} \|\nabla\psi(z)\| = 0, \tag{15}$$

(iii) *u is a strongly evanescent solution of* (DS-2) *if and only if*

$$\inf_{t\geq 0} \psi(u(t)) > -\infty. \tag{16}$$

Proof. Let $u(\cdot)$ be a global solution of (DS-1). This is obviously also a global solution of (DS-2) and satisfies $\|u'(t)\|^2 = 2V(u(t))$. Let us first assume that (15) holds. If $\nabla\psi(u(0)) = 0$, then $u(t) = u(0)$ for all $t \geq 0$ and $u(\cdot)$ is trivially w-evanescent. If $\nabla\psi(u(0)) \neq 0$, then $V(u(t)) = \frac{1}{2}\|\nabla\psi(u(t))\|^2 \neq 0$ for all $t \geq 0$ (cf. Remark 2.2), and hence for every $s \geq 0$ it holds that

$$\inf_{t\geq 0} V(u(t)) = \inf_{t\geq s} V(u(t)) = 0,$$

yielding again that $u(\cdot)$ is a w-evanescent solution of (DS-2). The converse is obvious, hence (i) is established.

Assertion (ii) follows directly from Lemma 2.1(ii). ∎

Combining Lemmas 2.3 and 2.11 we get the following corollary.

COROLLARY 2.12. *Any bounded maximal solution of* (DS-1) *is a strongly evanescent solution of* (DS-2).

Combining Corollary 2.4 with Lemma 2.11 we obtain the following result.

PROPOSITION 2.13. *Let $\psi \in \mathscr{C}^2(\mathcal{H})$ be bounded from below and let $V(x) := \frac{1}{2}||\nabla\psi(x)||^2$. Then, for every $x_0 \in \mathcal{H}$,* (DS-2) *has at least one strongly evanescent solution satisfying $v(0) = x_0$ which coincides with the unique global solution of the first-order system* (DS-1).

3. Main results.

This section contains the main results of the manuscript, which are presented in three subsections. Before we proceed, let us first recall the following continuous form of the classical Opial lemma [33] which will be used in what follows. (See also [3, Lemma 17.2.5, Page 704] for a proof.)

LEMMA 3.1 (Opial-type lemma). *Let S be a nonempty subset of a Hilbert space H and let $w : [0, +\infty) \to \mathcal{H}$ be a map. Assume that for every $z \in S$ the limit $\lim_{t\to+\infty} \|w(t) - z\|$ exists and is finite and that all weak sequential limits of $w(\cdot)$, as $t \to +\infty$, belong to S. Then $w(t)$ converges weakly to a point of S as $t \to +\infty$.*

3.1. The ψ convex case.

Throughout this subsection we shall assume that the function $\psi \in \mathscr{C}^2(\mathcal{H})$ is convex and V is given by (1). We shall be interested in comparing the solutions of (DS-1) and (DS-2). The following result is essentially known (see for instance [12, Theorems 3.1 and 3.2] for a proof in the more general context of multivalued evolution equations).

PROPOSITION 3.2 (Lyapunov functions for (DS-1)). *Let $\psi \in \mathscr{C}^1(\mathcal{H})$ be convex. Then, for every initial condition $x_0 \in \mathcal{H}$, the unique maximal solution $u(\cdot)$ of* (DS-1) *satisfying* (I_0) *is global. Moreover,*

(i) *$\rho(t) = \psi(u(t))$ is convex, nonincreasing, and*

$$\inf_{t\geq 0} \psi(u(t)) = \lim_{t\to+\infty} \psi(u(t)) = \inf_{z\in\mathcal{H}} \psi(z); \tag{17}$$

(ii) *for every $y \in \mathcal{H}$ and $t > 0$ it holds that*

$$\|u'(t)\| \leq \|\nabla\psi(y)\| + \frac{1}{t}\|u(0) - y\|;$$

(iii) *$t \mapsto \|u'(t)\| = \|\nabla\psi(u(t))\|$ is nonincreasing and*

$$\lim_{t\to+\infty} \|u'(t)\| = \inf_{z\in\mathcal{H}} \|\nabla\psi(z)\|; \tag{18}$$

(iv) *$\|u(\cdot) - \hat{x}\|$ is nonincreasing for every $\hat{x} \in \mathrm{Crit}_\psi$.*

Remark 3.3 (energy function). Under the assumptions of Proposition 3.2, for every $y \in \mathcal{H}$ we set

$$E_y(t) := \frac{1}{2}\|u(t)-y\|^2 + \int_0^t (\psi(u(s)) - \psi(y))\, ds.$$

Since ψ is convex, we deduce $E_y'(t) = \langle \nabla\psi(u(t)) \mid y - u(t)\rangle + \psi(u(t)) - \psi(y) \le 0$, that is, $E_y(\cdot)$ is nonincreasing on $[0, +\infty)$.

The following proposition is well known. It relates the behaviour of the orbits to the critical points of ψ. In what follows we set

$$\operatorname{dist}(x, \operatorname{Crit}_\psi) := \inf_{y \in \operatorname{Crit}_\psi} \|x - y\|.$$

PROPOSITION 3.4. *Let $\psi \in \mathscr{C}^1(\mathcal{H})$ be convex and $u(\cdot)$ be a global solution of* (DS-1).

(i) *If $\operatorname{Crit}_\psi \neq \emptyset$, then $\lim_{t\to+\infty}\|u'(t)\| = 0$ and there exists $\hat{x}_\star \in \operatorname{Crit}_\psi$ such that $u(t) \xrightarrow[t\to+\infty]{} \hat{x}_\star$ (weakly). Moreover, $\rho_*(t) := \psi(u(t)) - \psi(\hat{x}_\star) \in L^1(0, +\infty)$ and*

$$\int_0^{+\infty} (\psi(u(s)) - \min\psi)\ ds \ \le\ \frac{1}{2}\operatorname{dist}(u(0), \operatorname{Crit}_\psi)^2. \tag{19}$$

(ii) *If $\operatorname{Crit}_\psi = \emptyset$, then $\lim_{t\to+\infty}\|u(t)\| = +\infty$.*

(iii) *$u(\cdot)$ is bounded if and only if $\operatorname{Crit}_\psi \neq \emptyset$.*

Proof. The first part of assertion (i) follows from [3, Theorem 17.2.7]. Fix now any $\hat{x}_\star \in \operatorname{Crit}_\psi$. Since $E_{\hat{x}_\star}(t) \le E_{\hat{x}_\star}(0)$ and $\psi(\hat{x}_\star) = \min\psi$, taking the limit as $t \to +\infty$ we deduce

$$\int_0^{+\infty} (\psi(u(s)) - \min\psi)\ ds \ \le\ \frac{1}{2}\|u(0) - \hat{x}_\star\|^2. \tag{20}$$

Taking the infimum in (20) for $\hat{x}_\star \in \operatorname{Crit}_\psi$, we obtain (19).

Assertion (ii) follows from [3, Corollary 17.2.1], while assertion (iii) is a straightforward consequence of the last two assertions. ∎

The following result will play a key role in what follows.

PROPOSITION 3.5. *Let $\psi \in \mathscr{C}^2(\mathcal{H})$ be convex and $V(x) = \frac{1}{2}\|\nabla\psi(s)\|^2$. Then any w-evanescent solution of* (DS-2) *is also a (global) solution of the gradient system* (DS-1).

Proof. Let $v(\cdot)$ be a w-evanescent solution of (DS-2) and set $\phi(t) = v'(t) + \nabla\psi(v(t))$. Then, for all $t \ge 0$, it holds that $\|\phi(t)\| \le \|v'(t)\| + \|\nabla\psi(v(t))\|$. By Lemma 2.7 we deduce

$$\|\phi(t)\| \le 2\|\nabla\psi(v(t))\| = 2\|v'(t)\| \to 0,$$

whence $\liminf_{t\to+\infty}\|\phi(t)\| = 0$, since $v(\cdot)$ is a w-evanescent solution. We also know that

$$\phi'(t) = \nabla^2\psi(v(t))\phi(t)$$

(see Proposition 2.9(iv)). Thus,

$$\frac{d}{dt}\left(\|\phi(t)\|^2\right) = 2\langle \phi(t) \mid \nabla^2\psi(v(t))\phi(t)\rangle \ge 0,$$

since ψ is convex. Hence, $\|\phi\|^2$ is increasing. Therefore, since $\liminf_{t\to+\infty}\|\phi(t)\| = 0$ we deduce $\phi = 0$, which yields that $v(\cdot)$ is a solution of the first-order gradient system (DS-1). ∎

We are now ready to state our main results.

THEOREM 3.6 (second-order gradient system; ψ convex). *If $\psi \in \mathscr{C}^2(\mathcal{H})$ is convex,* (DS-2) *has a w-evanescent solution $v(\cdot)$ satisfying* (I_0) *if and only if (C) holds. Then $v(\cdot)$ is unique and is also the unique solution of the first-order system* (DS-1) *which satisfies* (I_0)*. Moreover,*

(i) *v is strongly evanescent if and only if ψ is bounded below,*
(ii) *v is bounded if and only if* $\mathrm{Crit}_\psi \neq \emptyset$.

Proof. As already mentioned in the introduction, condition (C) is necessary for the existence of a w-evanescent solution of (DS-2). Conversely, suppose that (C) is fulfilled. Then there exists a unique global solution $u(\cdot)$ of (DS-1) satisfying $u(0) = u_0 \in \mathcal{H}$ (cf. Proposition 3.2). Condition (15) is fulfilled thanks to (18) and (C). Thus, in view of Lemma 2.11, $u(\cdot)$ is a w-evanescent solution of (DS-2) satisfying (I_0). Uniqueness is straightforward from Proposition 3.5. Indeed, any w-evanescent solution of (DS-2) which satisfies (I_0) is necessarily the unique global solution of (DS-1) under the same initial condition (I_0). Finally, combining (16) with (17) we deduce that this solution is strongly evanescent if and only if ψ is bounded below. From Proposition 3.2, we also deduce that this solution is bounded if and only if $\mathrm{Crit}_\psi \neq \emptyset$. ∎

To illustrate Theorem 3.6 consider the convex $\mathscr{C}^2$ function ψ given in (2). Recall that ψ satisfies (C) but not $(C^\star)$. The first-order system $u'(t) = -\psi'(u(t))$, $u(0) = 0$ has the unique solution $u(t) = 1 - \sqrt{1+2t}$, $t \geq 0$, which is also the unique w-evanescent solution of (DS-2) (cf. Theorem 3.6). Clearly this solution is not strongly evanescent (ψ is not bounded from below).

An immediate consequence of Theorem 3.6 and Proposition 3.2 is the following result.

COROLLARY 3.7. *Let $\psi \in \mathscr{C}^2(\mathcal{H})$ be convex, assume (C) holds and let $v(\cdot)$ be a w-evanescent solution of* (DS-2)*. Then $v(\cdot)$ satisfies the properties stated in Proposition* 3.2 *and Proposition* 3.4.

We are ready to state the following surprising consequence.

THEOREM 3.8 (determination via modulus of gradient). *Let $\psi_1, \psi_2 \in C^2(\mathcal{H})$ be convex and assume*

– $\|\nabla\psi_1(z)\| = \|\nabla\psi_2(z)\|$ *for all $z \in \mathcal{H}$,*
– $\inf_{z\in\mathcal{H}} \|\nabla\psi_1(z)\| = 0$ *(this assumption holds whenever ψ_1 or ψ_2 is bounded below).*

Then, $\psi_1 = \psi_2 + c$ for some constant $c \in \mathbb{R}$.

Proof. Let ψ_1 and ψ_2 be two convex functions of class $\mathscr{C}^2$ satisfying $\|\nabla\psi_1\| = \|\nabla\psi_2\|$ and $\inf_{z\in\mathcal{H}}\|\nabla\psi_1(z)\| = 0$. Let $x \in \mathcal{H}$ be an arbitrary point and let $v(\cdot)$ be the unique weakly evanescent solution of the system

$$v''(t) = \nabla V(v(t)) \quad \text{for } t \geq 0,\ v(0) = x,$$

with $V = \frac{1}{2}\|\nabla\psi_1\|^2 = \frac{1}{2}\|\nabla\psi_2\|^2$ (cf. Theorem 3.6). Then $v(\cdot)$ is also a solution of the first-order systems

$$v'(t) = -\nabla\psi_1(v(t)), \quad v(0) = x,$$

and

$$v'(t) = -\nabla\psi_2(v(t)), \quad v(0) = x.$$

Hence, $\nabla\psi_1(x) = \nabla\psi_2(x)$. Since x is arbitrary, the result follows. ∎

Remark 3.9. In [6] it has been shown that a continuous (respectively, smooth) convex 1-coercive function can be determined (up to a constant) by knowing its subgradients (respectively, gradients) in specific points of its domain (namely, the ones that correspond to strongly exposed points of the epigraph). Theorem 3.8 asserts that a knowledge of the modulus of the gradient (rather than the gradient itself) suffices to determine a $\mathscr{C}^2$ convex function, provided the function is bounded from below.

Remark 3.10. The assumption that ψ_1, ψ_2 are bounded from below is important. Consider for instance the example of the functions $\psi_1(x) = x$ and $\psi_2(x) = -x$.

A direct consequence of Theorem 3.8 is the following result concerning uniqueness of convex $\mathscr{C}^2$-smooth solutions of the forthcoming eikonal equation (21).

COROLLARY 3.11 (eikonal equation I). *Let $f \in \mathscr{C}^1(\mathcal{H})$ be nonnegative. Then, the eikonal equation*

$$\|\nabla\psi\|^2 = f \tag{21}$$

has at most one (up to a constant) convex, bounded below solution in $\mathscr{C}^2(\mathcal{H})$.

Remark 3.12. The above result might appear to be restrictive at a first sight. Indeed, solving (21) in a viscosity sense leads to the existence of possibly nonsmooth solutions. In particular, if $\mathcal{H} = \mathbb{R}^d$ and $f(x) \geq \alpha > 0$, for all $x \in \mathbb{R}^d$, then any viscosity solution of (21) is unbounded from below (see, e.g., [10, Theorem 1.1]). Nonetheless, the case in which f is nonnegative and vanishes is actually of great interest for establishing some weak KAM theorems or existence of solutions for ergodic problems associated with first-order Hamilton–Jacobi equations. It is also known that (21) may have essentially different solutions; see [30] or [29]. See also [32] and references therein for the periodic case, and [9, 21, 10] for the unbounded case. In the above framework, the set of solutions of (21) is a challenging issue. The above result as well as the forthcoming Corollary 3.21 could eventually shed new light on this intriguing issue.

Before we finish this section, let us observe that the assumption $\psi_1, \psi_2 \in \mathscr{C}^2(\mathcal{H})$ in Theorem 3.8, although required in this approach (in view of (DS-2)), does not seem to be indispensable for the validity of the result. Indeed the conclusion of Theorem 3.8 also seems plausible for C^1-convex functions, or even for (nonsmooth) convex continuous functions, under a different approach. We propose below the following conjecture which, if true, would generalize Theorem 3.8.

Conjecture 3.13. Let $\psi_1, \psi_2 : \mathcal{H} \to \mathbb{R}$ be two (finite) convex functions bounded from below such that

$$\inf_{p\in\partial\psi_1(x)} ||p|| = \inf_{q\in\partial\psi_2(x)} ||q|| \quad \text{for all } x \in \mathcal{H}. \tag{22}$$

Then $\psi_1 = \psi_2 + c$ for some constant $c > 0$.

Proof of the conjecture if $\mathcal{H} = \mathbb{R}$. Let us denote by D the set of points where both ψ_1 and ψ_2 are simultaneously differentiable. Then ψ_1', ψ_2' are increasing functions on D and D is dense in $\mathbb{R}$. It follows from (22) that $\psi_1'(x) = \sigma(x)\psi_2'(x)$ for all $x \in D$, where $\sigma(x) \in \{-1, 1\}$. Our task is to establish that $\sigma \equiv 1$, that is, $\psi_1' = \psi_2'$ on D.

Then, since ψ_1, ψ_2 are locally Lipschitz (hence absolutely continuous), the conclusion follows.

Notice that (22) yields that ψ_1, ψ_2 have a common set of global minimizers. Denote by $S = \arg\min \psi_1 = \arg\min \psi_2$ this set. If $S = \emptyset$, then it is easily seen that the set of all subgradients $\partial\psi_i(\mathbb{R}) = \cup_{x\in\mathbb{R}}\partial\psi_i(x)$ is either contained in $(-\infty, 0)$ or in $(0, +\infty)$ for $i \in \{1, 2\}$. If $\partial\psi_1(\mathbb{R})$ is contained in $(-\infty, 0)$ and $\partial\psi_2(\mathbb{R})$ is contained in $(0, +\infty)$, then we would have $\sigma \equiv -1$ and $\psi_1' = -\psi_2'$ on D. Taking into account that ψ_1', ψ_2' are increasing, we deduce that ψ_1', ψ_2' are constant, which is impossible since $S = \emptyset$ and ψ_1, ψ_2 are bounded from below. Therefore both $\partial\psi_1(\mathbb{R})$ and $\partial\psi_2(\mathbb{R})$ are contained in the same interval $(-\infty, 0)$ or $(0, +\infty)$ and $\sigma \equiv 1$.

Consider now the $S \neq \emptyset$ case. If $S = \mathbb{R}$, then ψ_1, ψ_2 are constant and the result holds trivially, while for any $x > \sup S$ (respectively, any $x < \inf S$) we should have $\partial\psi_i(x) \subset (0, +\infty)$ (respectively, $\partial\psi_i(x) \subset (-\infty, 0)$) by monotonicity. Therefore again $\sigma \equiv 1$ and the conclusion follows. ∎

3.2. The V convex case. In this subsection the driving assumption is the convexity of the function $V(x) = \frac{1}{2}||\nabla\psi(x)||^2$, where $\psi \in \mathscr{C}^2(\mathcal{H})$. The focus is again the comparison of the solutions of the systems (DS-1) and (DS-2).

The following result reveals a characteristic property of the solutions of (DS-2), which is reminiscent of an analogous property for the orbits of the first-order system with convex potential.

PROPOSITION 3.14 (contraction of solutions of (DS-2)). *Let $\psi \in \mathscr{C}^2(\mathcal{H})$ and assume that $V(x) = \frac{1}{2}||\nabla\psi(x)||^2$ is convex. If v_1 and v_2 are two strongly evanescent solutions of* (DS-2)*, then the function*

$$q(t) := \frac{1}{2}\left\|v_1(t) - v_2(t)\right\|^2$$

is convex and nonincreasing on $[0, +\infty)$. In particular if $v_1(0) = v_2(0)$, then $v_1 = v_2$.

Proof. It suffices to prove that q is convex and nonincreasing. Differentiating twice and invoking the monotonicity of ∇V (see, e.g., [35, Chapter 2]) we get

$$\begin{aligned} q''(t) &= \langle v_1''(t) - v_2''(t) \mid v_1(t) - v_2(t)\rangle + \left\|v_1'(t) - v_2'(t)\right\|^2 \\ &= \langle \nabla V(v_1(t)) - \nabla V(v_2(t)) \mid v_1(t) - v_2(t)\rangle + \left\|v_1'(t) - v_2'(t)\right\|^2 \geq 0, \end{aligned}$$

which yields convexity of q. Let us prove that q is decreasing. By Proposition 2.10, we have

$$\int_0^\infty \frac{q(t)}{t^2+1}dt = \frac{1}{2}\int_0^\infty \frac{\left\|v_2(t) - v_1(t)\right\|^2}{t^2+1}dt < +\infty.$$

Suppose that there exists $t_0 > 0$ such that $q'(t_0) > 0$. Since q is convex, we would have

$$q(t) \geq q'(t_0)(t - t_0) + q(t_0) \quad \text{for all } t \geq t_0,$$

yielding

$$\int_0^\infty \frac{q(t)}{t^2+1}dt = +\infty, \quad \text{a contradiction.}$$

Hence, q is decreasing and the result follows. ∎

LEMMA 3.15. *Let $\psi \in \mathscr{C}^2(\mathcal{H})$, assume $V(x) = \frac{1}{2}\left\|\nabla\psi(x)\right\|^2$ is convex, and let $v(\cdot)$ be a strongly evanescent solution of* (DS-2). *If* $\mathrm{Crit}_\psi \neq \emptyset$, *then*

(i) *$h(t) := \|v(t) - \hat{x}\|$ is nonincreasing for every $\hat{x} \in \mathrm{Crit}_\psi$,*
(ii) *$v(\cdot)$ is bounded,*
(iii) *there exists $\hat{x}_\star \in \mathrm{Crit}_\psi$ such that $v(t) \xrightarrow[t\to+\infty]{} \hat{x}_\star$ (weakly).*

Proof. Let $v(\cdot)$ be a strongly evanescent solution of the system (DS-2) and pick any $\hat{x} \in \mathrm{Crit}_\psi$. Applying Proposition 3.14 for $u_1(t) = v(t)$ and $u_2(t) = \hat{x}$ for $t \geq 0$, we get (i). Since $\mathrm{Crit}_\psi \neq \emptyset$, (ii) is a direct consequence of (i). Finally, (iii) can be proved in a similar way as in Proposition 3.4, using Lemma 3.1 and the convexity of V. The details are left to the reader. ◻

PROPOSITION 3.16 (second-order gradient system; V convex). *Let us assume that $V(x) = \frac{1}{2}\left\|\nabla\psi(x)\right\|^2$ is convex and $\psi \in \mathscr{C}^2(\mathcal{H})$ is bounded from below. Then* (DS-2) *has a unique strongly evanescent solution satisfying* (I_0) *which is also the unique solution of* (DS-1) *that satisfies* (I_0).

Proof. From Corollary 2.4 and Cauchy–Lipschitz there exists a unique global solution of (DS-1) satisfying the initial condition (I_0). According to Lemma 2.11 this solution is also a strongly evanescent solution of (DS-2). Uniqueness follows from Proposition 3.14. ◻

We obtain the following corollary as a consequence.

COROLLARY 3.17 (convexity criterium). *Let $V(x) = \frac{1}{2}\|\nabla\psi(x)\|^2$ be convex and $\psi \in \mathscr{C}^2(\mathcal{H})$ be bounded below. Then, ψ is convex.*

Proof. Fix $z_1, z_2 \in \mathcal{H}$ and denote by $u_1(\cdot)$ and $u_2(\cdot)$ solutions of (DS-1) with z_1 and z_2 as initial data. Since u_1 and u_2 are also strongly evanescent solutions of (DS-2), we know that the function

$$q(t) = \frac{1}{2}\left\|u_1(t) - u_2(t)\right\|^2 \quad \text{for } t \geq 0$$

is decreasing (cf. Proposition 3.14). Thus,

$$0 \geq q'(t) = -\langle \nabla\psi(u_1(t)) - \nabla\psi(u_2(t)) \mid u_1(t) - u_2(t)\rangle,$$

or, equivalently,

$$\langle \nabla\psi(u_1(t)) - \nabla\psi(u_2(t)) \mid u_1(t) - u_2(t)\rangle \geq 0.$$

Taking the limit as $t \to 0$ we deduce that

$$\langle \nabla\psi(z_1) - \nabla\psi(z_2) \mid z_1 - z_2\rangle \geq 0,$$

which yields that ψ is convex (see, e.g., [35, Chapter 2]). ◻

Remark 3.18. Corollary 3.17 is false if ψ is not bounded below. Indeed, let $\psi(x) = x^3$. Then $V(x) = \frac{1}{2}|\psi'(x)|^2 = \frac{9}{2}x^4$ is convex, but ψ is not. Another two-dimensional example is $\psi(x_1, x_2) = x_1^4 - x_2^2$.

Remark 3.19. If $V(x) = \frac{1}{2}\left\|\nabla\psi(x)\right\|^2$ is convex and $\psi \in \mathscr{C}^2(\mathcal{H})$ is bounded from below, then combining Corollary 3.17 with Theorem 3.6 we deduce that every strongly evanescent solution $u(\cdot)$ of (DS-2) satisfies the assertions of Corollary 3.7 (since ψ is convex). In particular, $u(\cdot)$ is bounded if and only if $\mathrm{Crit}_\psi \neq \emptyset$.

The following result is a direct consequence of Theorem 3.8 and Corollary 3.17.

COROLLARY 3.20. *Let $\psi_1, \psi_2 \in \mathscr{C}^2(\mathcal{H})$ be bounded below satisfying*

$$\|\nabla\psi_1(x)\| = \|\nabla\psi_2(x)\| \quad \textit{for all } x \in \mathcal{H}.$$

Then, if $V(x) = \frac{1}{2}\|\nabla\psi_1(x)\|^2$ $(= \frac{1}{2}\|\nabla\psi_2(x)\|^2)$ is convex, we deduce that both ψ_1 and ψ_2 are convex and equal (up to a constant).

An illustration of Corollary 3.20 is given in case the case in which ψ_1 and ψ_2 are of the quadratic form

$$\psi_1(x) = \frac{1}{2}\langle x \mid A_1 x\rangle \quad \text{and} \quad \psi_2(x) = \frac{1}{2}\langle x \mid A_2 x\rangle,$$

where A_i is a symmetric linear bounded operator for $i \in \{1, 2\}$. One can quickly check that ψ_i is bounded below if and only if A_i is positive semidefinite. In the latter case, the identity $\|\nabla\psi_1\| = \|\nabla\psi_2\|$ means that $\|A_1 x\| = \|A_2 x\|$ for all $x \in \mathcal{H}$, yielding $A_1^2 = A_2^2$. Thus, $A_1 = A_2$ (since A_1 and A_2 are positive semidefinite) and $\psi_1 = \psi_2$. This is in accordance with Corollary 3.20. This example also shows the importance of the assumption that ψ_1 and ψ_2 are bounded below. Indeed, if $A_2 = -A_1 \neq 0$, then $\|\nabla\psi_1\| = \|\nabla\psi_2\|$ and $\psi_1 - \psi_2$ is not constant.

A direct consequence of Corollary 3.20 is the following result.

COROLLARY 3.21 (eikonal equation II). *Let $f \in \mathscr{C}^1(\mathcal{H})$ be nonnegative and convex. Then, the eikonal equation*

$$\|\nabla\psi\|^2 = f \tag{23}$$

has at most one bounded below solution in $\mathscr{C}^2(\mathcal{H})$ up to an additive constant. In addition, this solution is convex.

3.3. An alternative variational principle for (DS-1). In [14, 12, 13], Brézis and Ekeland proved the following variational characterization when ψ is a proper, convex, and lower semicontinuous functional defined on a Hilbert space $\mathcal{H}$. In this case (DS-1) becomes

$$u'(t) \underset{\text{a.e}}{\in} -\partial\psi(u(t)), \quad t \geq 0.$$

If $u(\cdot)$ is an absolutely continuous solution of the above differential inclusion on $[0, T]$ for some $T > 0$, with initial condition (I_0), then $u(\cdot)$ is the unique minimizer of the functional

$$\mathscr{J}(u) = \int_0^T \left(\psi(u(t)) + \psi^\star(-u'(t))\right) dt + \frac{1}{2}\left\|u(T)\right\|^2,$$

where $\psi^\star$ designates the Legendre conjugate of ψ. We also refer to [5] and [22] for extensions of this variational principle.

We now present an alternative variational principle for the first-order gradient system (DS-1). The formulation is based on the connection with the second-order system (DS-2). This latter can be seen as the Euler–Lagrange equation associated with a conventional functional. More precisely, for any real number $T > 0$, we consider the functional

$$J(T; w) = \int_0^T \left(\frac{1}{2}\left\|w'(t)\right\|^2 + \frac{1}{2}\|\nabla\psi(w(t))\|^2\right) dt + \psi(w(T)).$$

We state the following proposition.

PROPOSITION 3.22 (variational formulation). *Let* $V(x) = \frac{1}{2}\|\nabla\psi(x)\|^2$ *be convex,* $\psi \in \mathscr{C}^2(\mathcal{H})$ *be bounded below, and* $T > 0$*. Then* $u \in \mathscr{C}^0([0,T];\mathcal{H}) \cap \mathscr{C}^1((0,T);\mathcal{H})$ *is a solution of* (DS-1) *on* $[0,T]$ *if and only if*

$$J(T;u) \leq J(T;w) \tag{24}$$

for all $w \in \mathscr{C}^0([0,T];\mathcal{H}) \cap \mathscr{C}^1((0,T);\mathcal{H})$ *satisfying* $w(0) = u(0)$*.*

Proof. In view of Corollary 3.17, ψ is convex. Let $u(\cdot)$ be a solution of (DS-1) on $[0,T]$ and $w \in C^1([0,T],\mathcal{H})$ be such that $w(0) = u(0)$. Set $h = w - u$. Then,

$$J(T;w) - J(T;u) = \int_0^T \left(\langle u'(t) \mid h'(t)\rangle + \frac{1}{2}\|h'(t)\|^2 + V(u(t)+h(t)) - V(u(t)) \right) dt \\ + \psi(u(T)+h(T)) - \psi(u(T)).$$

Using convexity of ψ and V we deduce

$$J(T;w) - J(T;u) \geq \int_0^T \left(\langle u'(t) \mid h'(t)\rangle + \langle \nabla V(u(t)) \mid h(t)\rangle \right) dt \\ + \langle \nabla\psi(u(T)) \mid h(T)\rangle.$$

Integrating by parts yields

$$J(T;w) - J(T;u) \\ \geq \int_0^T \langle -u''(t) + \nabla V(u(t)) \mid h(t)\rangle\, dt + \langle u'(T) + \nabla\psi(u(T)) \mid h(T)\rangle.$$

Since $u(\cdot)$ is solution of (DS-1), it is also solution of (DS-2), and therefore

$$\int_0^T \langle -u''(t) + \nabla V(u(t)) \mid h(t)\rangle\, dt + \langle u'(T) + \nabla\psi(u(T)) \mid h(T)\rangle = 0,$$

yielding $J(T;w) \geq J(T;u)$.

Conversely, let $u \in C^1([0,T],\mathcal{H})$. Assume that $J(T;w) \geq J(T;u)$ for all $w \in C^1([0,T],\mathcal{H})$ such that $w(0) = u(0)$. By a conventional argument, we know that u is of class $\mathscr{C}^2$. Moreover, u satisfies the Euler–Lagrange equation $u''(t) = \nabla V(u(t))$ and the *transversality* condition $u'(T) + \nabla\psi(u(T)) = 0$. Set $\phi(t) = u'(t) + \nabla\psi(u(t))$ for $t \geq 0$. We know that ϕ is a solution of the linear differential equation $\phi'(t) = \nabla^2\psi(u(t))\phi(t)$ (see Proposition 2.9) with $\phi(T) = 0$, so then ϕ is the trivial solution $\phi = 0$, that is, u is a solution of (DS-1) on $[0,T]$. This ends the proof. ∎

We now consider the functional

$$J_\infty^\star(w) = \int_0^{+\infty} \left(\frac{1}{2}\|w'(t)\|^2 + \frac{1}{2}\|\nabla\psi(w(t))\|^2 \right) dt.$$

We also state the following corollary.

COROLLARY 3.23. *Suppose that* $\psi \in \mathscr{C}^2(\mathcal{H})$ *is bounded below,* $\mathrm{Crit}_\psi \neq \emptyset$*, and* $[x \mapsto \|\nabla\psi(x)\|^2]$ *is convex. Then,* $u \in \mathscr{C}^1([0,+\infty);\mathcal{H})$ *is a global solution of* (DS-1) *if and only if*

$$J_\infty(u) \leq J_\infty(w) \tag{25}$$

for any bounded function $w \in \mathscr{C}^1([0,+\infty);\mathcal{H})$ *with* $w(0) = u(0)$*.*

Proof. Let $u \in \mathscr{C}^1([0,+\infty);\mathcal{H})$ be a global solution of (DS-1). For $T > 0$ and $z \in \mathscr{C}^1([0,T];\mathcal{H})$ we set

$$J^\star(T;z) = \int_0^T \left(\|z'(t)\|^2 + \|\nabla\psi(z(t))\|^2\right) dt.$$

Let $w \in \mathscr{C}^1([0,+\infty);\mathcal{H})$ be a bounded function satisfying $w(0) = u(0)$ and set $h = w - u$. Following the proof of inequality (24), we obtain

$$J^\star(T;w) \geq J^\star(T;u) + \langle u'(T) \mid h(T)\rangle. \tag{26}$$

Let us observe that u is bounded and $\lim_{T\to+\infty} \|u'(T)\| = 0$ (thanks to Proposition 3.4). Thus, $h = w - u$ is also bounded. Taking the limit when $T \to +\infty$ yields

$$J^\star_\infty(w) \geq J^\star_\infty(u). \tag{27}$$

Conversely, suppose that $u \in \mathscr{C}^1([0,+\infty);\mathcal{H})$ satisfies (27) for any bounded function $w \in \mathscr{C}^1([0,+\infty);\mathcal{H})$ with $w(0) = u(0)$. Let $\hat{x} \in \mathrm{Crit}_\psi \neq \emptyset$ and consider the function

$$w_0(t) = e^{-t}(u(0) - \hat{x}) + \hat{x}.$$

Let us denote by $[\hat{x}, u(0)] = \{\theta(u(0) - \hat{x}) + \hat{x}\,;\, \theta \in [0,1]\}$ the segment between $\hat{x}$ and $u(0)$. Obviously $[\hat{x}, u(0)]$ is a compact subset of $\mathcal{H}$ and $w_0(t) \in [\hat{x}, u(0)]$ for all $t \geq 0$. We deduce that

$$V(w_0(t)) = V(w_0(t)) - V(\hat{x}) \leq \sup_{x\in[\hat{x},u(0)]} \|\nabla V(x)\| \, \|w_0(t) - \hat{x}\|\,.$$

It follows that $V(w_0(t)) \leq \sup_{x\in[\hat{x},u(0)]} \|\nabla V(x)\| \, \|w_0(0) - \hat{x}\| \, e^{-t}$. Therefore we obtain $J^\star_\infty(w_0) < +\infty$, whence $J^\star_\infty(u) < +\infty$.

Consider now an arbitrary real number $T > 0$ and let $h \in \mathscr{C}^1([0,+\infty);\mathcal{H})$ have a compact support included in $[0,T]$. Then,

$$J^\star_\infty(u+h) - J^\star_\infty(u) = J^\star(T;u+h) - J^\star(T;u).$$

Thus,

$$J^\star_T(u+h) \geq J^\star_T(u).$$

From the latter we deduce that u satisfies the Euler–Lagrange equation $u''(t) = \nabla V(u(t))$ on $(0,T)$. Since $T > 0$ is arbitrary, u is a global solution of (DS-2) on $[0,+\infty)$. Since $J^\star_\infty(u) < +\infty$, it is also a strongly evanescent solution. In view of Proposition 3.16, u is also solution of (DS-1). ∎

Remark 3.24. In the second part of the proof, we can show that $J^\star_\infty(u) < +\infty$ in another way. Indeed, one can choose w_0 as the unique strongly evanescent solution of (DS-2) which satisfies $w_0(0) = u(0)$ (existence of w_0 is ensured by Proposition 3.16). In view of Remark 3.19, we know that w_0 is bounded.

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