Integration of multivalued operators and cyclic submonotonicity

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Abstract We introduce a notion of cyclic submonotonicity for multivalued operators from a Banach space X to its dual. We show that if the Clarke subdifferential of a locally Lipschitz function is strictly submonotone on an open subset U of X, then it is also maximal cyclically submonotone on U, and, conversely, that every maximal cyclically submonotone operator on U is the Clarke subdifferential of a locally Lipschitz function, which is unique up to a constant if U is connected. In finite dimensions these functions are exactly the lower-C^1 functions considered by Spingarn and Rockafellar.

Key words Integration, subdifferential, submonotone operator, subsmooth function.

AMS Subject Classification *Primary* 49J52; 47H05; *Secondary* 58C20.

1 Introduction

We deal with the integration of a multivalued operator considered as the inverse process of taking the subdifferential of a function. This important question has been tackled by several authors: see for instance [20] (for the Fenchel-Moreau subdifferential of a convex function), [12], [19], [17] (for the Clarke subdifferential in finite dimensions), [4], [26] (for the Clarke subdifferential in infinite dimensions), [1] for the moderate subdifferential of Michel-Penot in finite dimensions) and [25], [27] (for various subdifferentials of a lower semicontinuous function). The first mentioned result concerns the case of monotone operators: in [20] Rockafellar shows that any cyclically monotone operator T is included in the subdifferential of a lower semicontinuous convex function, with equality if T is maximal cyclically monotone. Janin [12] introduces a concept of cyclic submonotonicity in finite dimensional Euclidean spaces and uses it to integrate (in the preceding sense) locally bounded operators satisfying that condition into locally Lipschitz functions. In a different context, a concept of cyclicity has also been used by Qi in [19], where the author characterizes operators that coincide with a subdifferential of some locally Lipschitz function, using the Lebesgue measure and (implicitly) the Rademacher theorem. Elaborating these ideas, Borwein and Moors [3] introduce and study the class $S_e(X)$ of *essentially smooth* (locally Lipschitz) functions, that is functions f whose Clarke subdifferential $\partial^C f$ is single-valued in the complement of a Haar null set. One of the main features of this class stems from the fact that for every $f \in S_e(X)$, the problem of finding a locally Lipschitz function g such that $\partial^C g \subseteq \partial^C f$ has a unique solution modulo a constant (i.e. $g = f + c$). In [4], Borwein, Moors & Shao extend the results of Qi ([19]) to separable Banach spaces,

using line integrals and Christensen's generalization of the Rademacher theorem (via Haar null sets). Their result is further improved by Wang [26], who characterizes the class of integrable locally bounded operators in separable Banach spaces. These operators are called in [19] and [26] "cyclically normal".

In another line of research, Spingarn [24] showed that in finite dimensions, *lower C^1* functions (i.e. functions arising as maxima of compactly indexed families of C^1 functions) are characterized by the fact that their Clarke subdifferentials are *strictly submonotone* operators with non-empty values. This last notion was extended in infinite dimensions by Georgiev in [10], [11] (see definition of *strict directional submonotonicity* in Section 2). Functions with such subdifferentials (hereby called *subsmooth*) are always regular (in the sense of Clarke [6]) and semi-smooth (in the sense of Mifflin [14]), see [24] and [11]. In particular, subsmooth functions have "small" (namely *minimal w^*-cusco*) Clarke subdifferentials. Let us recall that, in general, Lipschitz functions have "generically" very large Clarke subdifferentials ([26]). In finite dimensions, as the notions of strict monotonicity and strict directional submonotonicity coincide [11], a function f is subsmooth if, and only if, it is lower C^1.

Our main results rely on a notion of cyclic submonotonicity introduced here; in finite dimensions it coincides with the definition of Janin [12]. Using this concept we show that if U is an open subset of a Banach space X, then

- the subdifferential of every subsmooth function (defined on U) is maximal cyclically submonotone (on U).

- if U is connected and f, g are subsmooth functions on U such that $\partial^C f = \partial^C g$, then $f = g + c$ for some constant $c \in \mathbb{R}$.

- every maximal cyclically submonotone operator on U is the subdifferential of a subsmooth function defined on U (unique up to a constant if U is connected).

A specific feature of our approach is that it does not depend on results from measure theory and is valid beyond the class of separable spaces.

Notation Let us now fix our notation. We denote by $(X, \|.\|)$ a Banach space, by S_X its unit sphere and by $(X^*, \|.\|)$ its dual space. We also denote by $B_r(x)$ (resp. $B_r[x]$) the open (resp. closed) ball with center x and radius r and by $B_r^*(x)$ (resp. $B_r^*[x]$) the same objects in X^*. Let 2^{X^*} be the set of all subsets of X^*, $\mathbb{R}$ (resp. $\mathbb{N}$) be the set of all real (resp. non negative integer) numbers. For any $k \in \mathbb{N}$ we set $\mathbb{N}_k := \{1, 2, ..., k\}$. For any x, y in X we denote by $[x, y] := \{x_t = tx + (1 - t)y, 0 \leq t \leq 1\}$ the closed segment with endpoints x, y. For any subset K of X and any $\delta > 0$, we consider the δ- (open) neighborhood $B_\delta(K)$ of K defined by $B_\delta(K) := \{x \in X : \exists y \in K, \|x - y\| < \delta\}$. We also set $diam(K) := \sup\{\|x - y\| : x, y \in K\}$ for the diameter of the set K. We denote by $cone(K)$ the cone generated by K, i.e. $cone(K) := \{\lambda x : \lambda \geq 0, \ x \in K\}$. For any subset A of X^* we denote by $\overline{co}^{w^*}(A)$ the w^*-closed convex hull of A. Finally, throughout the paper we shall assume that *all functions f are locally Lipschitz* and we shall denote by $dom(f)$ their domain.

2 Preliminaries

Given a multivalued operator $T : X \rightrightarrows X^*$, we denote by $\mathrm{dom}(T)$ the set $\{x \in X : T(x) \neq \emptyset\}$ and by $\mathrm{Gr}(T) := \{(x, x^*) \in X \times X^* : x^* \in T(x)\}$ (or simply T if no confusion may arise) the graph of T. We also define the operators $\overline{co}^{w^*}(T)$ by

$$\overline{co}^{w^*}(T)(x) := \overline{co}^{w^*}(T(x)) \tag{1}$$

for all $x \in X$ and $\overline{T} : X \to 2^{X^*}$ by

$$x^* \in \overline{T}(x) \iff \begin{cases} \exists \{x_i\}_i \text{ in } X, \ \exists \{x_i^*\}_i \text{ in } X^* \colon x_i^* \in T(x_i) \\[2mm] x = \lim_i x_i, \qquad x^* = w^*\text{-}\lim_i x_i^* \end{cases} \tag{2}$$

where $\{x_i\}_i$ and $\{x_i^*\}_i$ denote respectively nets in X and in X^*. Note that $\mathrm{Gr}(\overline{T}) = \overline{\mathrm{Gr}(T)}^{\|\cdot\| \times w^*}$.

The operator $T : X \rightrightarrows X^*$ is said to be *locally bounded* at $x \in X$ if there exist $M > 0$ and a neighborhood $B_r(x)$ of x such that $u \in B_r(x)$ and $u^* \in T(u)$ imply $\|u^*\| < M$. Then T is called locally bounded on a subset U of X, if T is locally bounded at all $x \in U$. (Note that U is not necessarily a subset of $\mathrm{dom}(T)$).

Furthermore, the operator T is said to be w*-upper-semicontinuous at $x \in X$, if for every w*-open set $W \supset T(x)$ there exists an open ball $B_r(x)$ of x such that $T(u) \subseteq W$ for every $u \in B_r(x)$. Let us note that if T is locally bounded on U, then $\overline{T}$ (given by relation (2)) is w*-upper-semicontinuous at every $x \in \mathrm{dom}(\overline{T}) \cap U$.

w*-cusco mappings A multivalued mapping $T : X \rightrightarrows X^*$ is said to be *w*-cusco* on U ([2], [5] e.g.), if it is w*-upper semicontinuous with nonempty w*-compact convex values on U. A w*-cusco mapping on U that does not strictly contain any other w*-cusco mapping with domain in U is called a *minimal w*-cusco* on U.

Given an operator S, we can consider w*-cusco mappings T that are minimal under the property of containing S. In the important case of the following proposition, one can give a complete description of the minimal (in fact least) element of the family of w*-cusco mappings containing S.

Proposition 1 *Let S be a densely defined locally bounded operator on an open subset U of X with values in X^*. Then the family of w*-cusco mappings containing S has a least element T given by the formula*

$$T(x) = \overline{co}^{w^*} \bigcap_{\varepsilon > 0} \overline{\{S(x') : x' \in B_\varepsilon(x) \cap \mathrm{dom}(S)\}}^{w^*}. \tag{3}$$

Proof In [2, Proposition 1.3] (see also [15, Proposition 1.2]), the following formula for the operator T is given:

$$T(x) = \bigcap_{\varepsilon > 0} \overline{co}^{w^*} \{S(x') : x' \in B_\varepsilon(x) \cap \mathrm{dom}(S)\}.$$

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In order to justify (3), let us set

$$R(x) := \bigcap_{\varepsilon > 0} \overline{\{S(x') : x' \in B_\varepsilon(x) \cap \mathrm{dom}(S)\}}^{w^*}.$$

Since S is locally bounded on U, it is easily seen that $R(x) = \overline{S}(x)$ (given by relation (2)) and that R is the smallest w*-upper-semicontinuous multivalued mapping containing S. Thus $R(x) \subseteq T(x)$ and $\overline{co}^{w^*}(R(x)) \subseteq T(x)$ for each $x \in U$. Since $\overline{co}^{w^*}(R(x))$ is w*-cusco (see [5, Proposition 2.7] e.g.) and T is the minimal w*-cusco containing S, we get $\overline{co}^{w^*}(R) = T$. □

The preceding proposition has an interesting (and immediate) consequence upon the representation of the Clarke subdifferential in certain Banach spaces. We recall that the *Clarke generalized derivative* of a locally Lipschitz function f at a point $x \in \mathrm{dom}(f)$ is defined for all $u \in X$ as follows

$$f^o(x; u) = \limsup_{(y,t) \to (x, 0^+)} \frac{f(y + tu) - f(y)}{t}$$

and the *Clarke subdifferential* of f at $x \in \mathrm{dom}(f)$ by

$$\partial^C f(x) = \{x^* \in X^* : \langle x^*, u \rangle \leq f^o(x, u), \forall u \in X\}. \tag{4}$$

For all $x \in \mathrm{dom}(f)$ we have $\partial^C f(x) \neq \emptyset$.

Let us also recall the definitions of other usual subdifferentials that will occur in the sequel:

the *Fréchet subdifferential* $\partial^F f(x)$

$$\partial^F f(x) := \{x^* \in X^* : f(y) \geq f(x) + \langle x^*, y - x \rangle + o(y - x), \forall y \in X\}$$

where $o : X \to R$ is some real valued function satisfying $\lim_{u \to 0} \frac{o(u)}{\|u\|} = 0$;

the *Hadamard subdifferential* $\partial^H f(x)$

$$\partial^H f(x) = \{x^* \in X^* : \langle x^*, u \rangle \leq \liminf_{(w,t) \to (u, 0^+)} \frac{f(x + tw) - f(x)}{t}, \forall u \in X\}.$$

Let us note that if f is locally Lipschitz, then for all $u \in X$

$$df(x, u) := \liminf_{(w,t) \to (u, 0^+)} \frac{f(x + tw) - f(x)}{t} = \liminf_{t \searrow 0^+} \frac{f(x + tu) - f(x)}{t} := f'(x, u), \tag{5}$$

so that the Hadamard derivative coincides with the Gâteaux derivative of f when they exist.

Let us now recall that in every Asplund space, the Clarke subdifferential $\partial^C f$ of a locally Lipschitz function f is given by the following formula of Preiss ([18, Remark 2.3]):

$$\partial^C f(x) = \bigcap_{\varepsilon > 0} \overline{co}^{w^*} \{D^F f(x') : x' \in B_\varepsilon(x) \cap \mathrm{dom}(D^F f)\} \tag{6}$$

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while if X has a Gâteaux smooth renorming:

$$\partial^C f(x) = \bigcap_{\varepsilon > 0} \overline{co}^{w^*} \{D^H f(x') : x' \in B_\varepsilon(x) \cap \mathrm{dom}(D^H f)\} \qquad (7)$$

where $D^F f(x)$ (resp. $D^H f(x)$) denotes the Fréchet (resp. Hadamard) derivative of f at x and $\mathrm{dom}(D^F f)$ (resp. $\mathrm{dom}(D^H f)$) is the domain of $D^F f$ (resp. $D^H f$).

Since $\partial^C f$ is a w*-cusco mapping ([3]), combining Proposition 1 with formulas (6) and (7), we obtain in view of [18, Theorem 2.4] the following corollary.

Corollary 2 *For every (locally Lipschitz) function f on X we have*
(i) if X is an Asplund space then

$$\partial^C f(x) = \overline{co}^{w^*} \bigcap_{\varepsilon > 0} \overline{\{D^F f(x') : x' \in B_\varepsilon(x) \cap \mathrm{dom}(D^F f)\}}^{w^*} ; \qquad (8)$$

(ii) if X has an equivalent Gâteaux differentiable norm then

$$\partial^C f(x) = \overline{co}^{w^*} \bigcap_{\varepsilon > 0} \overline{\{D^H f(x') : x' \in B_\varepsilon(x) \cap \mathrm{dom}(D^H f)\}}^{w^*} . \qquad (9)$$

Submonotone and strictly submonotone mappings In 1981, J. Spingarn [24] introduced the notion of a *strictly submonotone* mapping in a finite dimensional space. His definition is naturally extended in infinite dimensions as follows: a multivalued mapping $T{:}X \rightrightarrows X^*$ is said to be *strictly submonotone* (in short *s-submonotone*) at $x \in X$ provided that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\frac{\langle x_1^* - x_2^*, x_1 - x_2 \rangle}{\|x_1 - x_2\|} \geq -\varepsilon \qquad (10)$$

whenever $x_i \in B_\delta(x)$, $x_i^* \in T(x_i)$, $i = 1, 2$ and $x_1 \neq x_2$.

The operator $T{:}X \rightrightarrows X$ is called *submonotone* at x, if (10) holds under the additional assumption $x_2 = x$. (Note that T is submonotone at every $x \notin \mathrm{dom}(T)$ and s-submonotone at every $x \notin \overline{\mathrm{dom}}(T)$.)

Appropriate directional versions of these notions have been introduced in [10] (see also [11] and [16]): an operator $T : X \rightrightarrows X^*$ is called *directionally strictly submonotone*[3] (in short *ds-submonotone*) at x, if for every $e \in S_X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\frac{\langle x_1^* - x_2^*, x_1 - x_2 \rangle}{\|x_1 - x_2\|} \geq -\varepsilon \qquad (11)$$

whenever $x_i \in B_\delta(x)$, $x_i^* \in T(x_i)$, $i = 1, 2$, $x_1 \neq x_2$ and $\|\frac{x_1 - x_2}{\|x_1 - x_2\|} - e\| < \delta$.

The operator $T : X \rightrightarrows X^*$ is called *directionally submonotone* (in short *d-submonotone*) at x, if (11) holds under the additional assumption $x_2 = x$.

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[3] "strictly submonotone" according to the terminology of [10], [11].

It is easily seen that if (11) holds uniformly for all directions $e \in S_X$, then T is s-submonotone at x. Similarly, if (11) holds uniformly for $x_2 = x$, then T is submonotone at x. If $X = \mathbb{R}^n$, the compactness of the unit sphere in $\mathbb{R}^n$ entails that an operator T is ds-submonotone (resp. d-submonotone) if, and only if, it is s-submonotone (resp. submonotone).

Given a nonempty subset U of X, we say that T is s-submonotone (resp. submonotone, ds-submonotone, d-submonotone) on U, if T has the corresponding property at every $x \in U$.

Let us recall from [11, Theorem 2.4] that every ds-submonotone operator T on X is *locally bounded* on $\operatorname{int} \operatorname{dom}(T)$. The definition of ds-submonotonicity (relation (11)) is reminiscent of monotonicity and can be considered as a limiting variant of it. It can also be considered as a mild continuity condition, since any continuous function $g : U \to X^*$ can be seen as a (single-valued) s-submonotone operator on U. Thus, every monotone operator is s-submonotone, while the converse is not true. The class of s-submonotone operators is stable under addition and is relatively large.

3   A sufficient condition for integration

In this section we give sufficient conditions for integrating multivalued operators. We first need some terminology. Given a segment $[x, y]$, a finite sequence $\{x_i\}_{i=1}^k$ of $[x, y]$ is called a *subdivision of the segment* $[x, y]$, if $x_1 = x$, $x_k = y$ and

$$\sum_{i=1}^{k-1} \|x_{i+1} - x_i\| = \|x - y\|. \tag{12}$$

A polygonal path $[w_h]_{h=1}^m$ is an union of consecutive segments; it is said to be closed if $w_m = w_1$. A finite sequence $\{x_i\}_{i=1}^n$ is called a *subdivision of the path* $[w_h]_{h=1}^m$, if there exists an increasing sequence $1 = k_1 < k_2 < ... < k_m = n$ such that for $1 \le h \le m - 1$, $\{x_i\}_{i=k_h}^{k_{h+1}}$ is a subdivision of the segment $[w_h, w_{h+1}]$.

The following definition is a reformulation in infinite dimensions of a property introduced by Janin ([12]) for the class of bounded operators defined on compact subsets of $\mathbb{R}^n$.

Definition 3 *An operator $T : X \rightrightarrows X^*$ is called radially cyclically submonotone on a subset U of its domain, if for any closed polygonal path $[w_h]_{h=1}^m \subseteq U$ and any $\varepsilon > 0$, there exists $\delta > 0$, such that for any subdivision $\{x_i\}_{i=1}^n$ of $[w_h]_{h=1}^m$ satisfying $\|x_{i+1} - x_i\| < \delta$ (for $i = 1, 2, ..., n$) and any $x_i^* \in T(x_i)$ one has:*

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \le \varepsilon. \tag{13}$$

The proof of the following result borrows ideas from [12], mainly on steps 1 and 4.

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Theorem 4 *Let $T : X \rightrightarrows X^*$ be locally bounded and radially cyclically submonotone on an open subset U of $\mathrm{dom}(T)$. Suppose that T is d-submonotone, or more generally, that for any $x \in U$, $x^* \in T(x)$, $u \in X$ one has*

$$\liminf_{t \to 0^+} \sup_{y^* \in T(x+tu)} \langle y^* - x^*, u \rangle \geq 0. \tag{14}$$

Then there exists a locally Lipschitz function $f : U \to \mathbb{R}$ such that $T \subseteq \partial^H f$ on U. If, in addition, T is submonotone, then $T \subseteq \partial^F f$ on U.

Proof

Case 1: Let us first suppose that U is connected. Then let V be the set of $(x,y) \in U \times U$ such that $[x,y] \subseteq U$. Given $(x,y) \in V$ and $\sigma > 0$, let us denote by $S_\sigma(x,y)$ the set of subdivisions $\{x_i\}_{i=1}^n$ of the segment $[x,y]$ such that $x_1 := x$, $x_n := y$ and $\|x_{i+1} - x_i\| < \sigma$, for $i = 1, 2, ..., n-1$.

We consider the function $g(\cdot, \cdot) : V \to \mathbb{R} \cup \{+\infty\}$ given by:

$$g(x,y) := \inf_{\sigma > 0} \sup \left\{ \sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle : \{x_i\}_{i=1}^n \in S_\sigma(x,y), \ x_i^* \in T(x_i) \right\}. \tag{15}$$

Since T is locally bounded, a compactness argument shows that for any $(x,y) \in V$ there exists $k > 0$ and $\rho > 0$ such that for all $(x', y') \in B_\rho(x) \times B_\rho(y)$, we have $(x', y') \in V$ and

$$|g(x', y')| \leq k\|x' - y'\|. \tag{16}$$

Let us now fix some x_0 in U and define $f : U \to \mathbb{R} \cup \{+\infty\}$ as follows:

$$f(x) := \sup \left\{ \sum_{h=1}^{m-1} g(w_h, w_{h+1}) \right\} \tag{17}$$

where the supremum is taken over all $m \geq 2$ and all polygonal paths $[w_h]_{h=1}^m$ with $w_1 := x_0$ and $w_m := x$ such that $[w_h, w_{h+1}] \subseteq U$, for all $h = 1, 2, ..., m-1$. (Note that this family of paths is non-empty as U is open and connected.)

Step 1: The domain of f is nonempty.

We shall show in particular that $f(x_0) = 0$. Since $f(x_0) \geq g(x_0, x_0) = 0$, it suffices to show that $f(x_0) \leq 0$. To this end, let us suppose that $f(x_0) > 0$ and take any $0 < \varepsilon < f(x_0)$. By (17) we infer that for some closed polygonal path $[w_h]_{h=1}^m$ (with $w_1 = w_m = x_0$) we have $\sum_{h=1}^{m-1} g(w_h, w_{h+1}) > \varepsilon$. Then according to (15), for any $\delta > 0$ we can find a subdivision $\{x_i\}_{i=1}^n$ of the path $[w_h]_{h=1}^n$ (where $x_1 = x_n = x_0$) and $\{x_i^*\}_{i=1}^n$ in X^* such that $x_i^* \in T(x_i)$, $\|x_{i+1} - x_i\| < \delta$ for all $i \geq 1$ and $\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle > \varepsilon$. Since T is radially cyclically submonotone, we get a contradiction.

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Step 2: $f(y) \geq f(x) + g(x,y), \quad \forall (x,y) \in V.$

Take any $r < f(x)$ and choose a polygonal path $[w_h]_{h=1}^m$ in U with $w_1 = x_0$ and $w_m = x$ such that $\sum_{h=1}^{m-1} g(w_h, w_{h+1}) > r$. Set $w_{m+1} = y$. It follows from (17) that

$$f(y) \geq \sum_{h=1}^m g(w_h, w_{h+1}) > r + g(w_m, w_{m+1}) = r + g(x,y).$$

Since $r < f(x)$ is arbitrarily close to $f(x)$, the proof is complete.

Step 3: $f(\bar{x}) < +\infty$, for all $\bar{x} \in U$ and f is locally Lipschitz on U.

Take any $\bar{x} \in U$ and choose a polygonal path $[w_h]_{h=1}^m$ in U with $w_1 = \bar{x}$ and $w_m = x_0$. It follows from Step 2 that for $y = x_0$ and for $x = w_{m-1}$ we have

$$0 = f(x_0) \geq f(w_{m-1}) + g(w_{m-1}, x_0)$$

which shows that $f(w_{m-1})$ is finite. Taking now $y = w_{m-1}$ and $x = w_{m-2}$ we conclude that $f(w_{m-2})$ is finite. Proceeding like this, we finally conclude that $f(\bar{x}) = f(w_1) < +\infty$. Now to show that f is locally Lipschitz, given $\bar{x} \in U$ we take $x = y = \bar{x}$ and $\rho > 0$ such that for any $x', y' \in B_\rho(\bar{x})$ we have $(x', y') \in V$ and the estimate in (16). It follows from Step 2 that

$$|f(y') - f(x')| \leq \max\{-g(x', y'), -g(y', x')\}$$

which yields that f is locally Lipschitz on U.

Step 4:

$$T(x) \subseteq \partial^H f(x) \quad \forall x \in U \tag{18}$$

Fix $x \in U$ and $x^* \in T(x)$. Let $r > 0$ be such that $B_r(x) \subseteq U$. For every $u \in S_X$, we have by Step 2 that

$$\frac{f(x+tu) - f(x)}{t} \geq \frac{g(x, x+tu)}{t}, \text{ for all } t \in \,]0, \, r[.$$

Since f is locally Lipschitz, it suffices to show that for any $u \in S_X$, $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\inf_{0 < t < \delta} \frac{g(x, x+tu)}{t} \geq \langle x^*, u \rangle - \varepsilon. \tag{19}$$

¿From (14), we can associate to any $\varepsilon > 0$ some $\delta > 0$ (depending on u) such that for all $s \in \,]0, \delta[$ we can find $y^* \in T(x + su)$ satisfying

$$\langle y^*, u \rangle \geq \langle x^*, u \rangle - \varepsilon. \tag{20}$$

Fix $0 < t < \delta$. Given $\sigma > 0$ and any subdivision $\{t_i\}_{i=1}^n$ of $[0, t]$ such that $\sup_i (t_{i+1} - t_i) < \sigma$ we can find $x_i^* \in T(x + t_i u)$ such that

$$\langle x_i^*, u \rangle \geq \langle x^*, u \rangle - \varepsilon.$$

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Setting $x_i := x + t_i u$, we observe that $\{x_i\}_{i=1}^n \in S_\sigma(x, x + tu)$ and

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle = \sum_{i=1}^{n-1} \langle x_i^*, (t_{i+1} - t_i)u \rangle$$

$$\geq (\langle x^*, u \rangle - \varepsilon) \sum_{i=1}^{n-1} (t_{i+1} - t_i) = \langle x^*, u \rangle t - \varepsilon t$$

Since $\sigma > 0$ can be taken arbitrarily small, we obtain

$$g(x, x + tu) \geq \langle x^*, tu \rangle - t\varepsilon$$

and relation (19) follows for δ introduced above.

Note that assuming in addition that T is submonotone, the above δ (in Step 4) does not depend on the direction $u \in S_X$, hence (19) yields $x^* \in \partial^F f(x)$.

Case 2: (general case) Let U be an arbitrary non-empty open set. Then U can be written as a disjoint union of open connected sets U_i. Applying the result of Case 1 for each i we obtain a locally Lipschitz function f_i on U_i with $T(x) \subseteq \partial^H f(x)$ (resp. $T(x) \subseteq \partial^F f(x)$, if T is submonotone), for all $x \in U_i$. Define $f : U \to \mathbb{R}$ by $f(x) = f_{i(x)}(x)$, where $i(x)$ is the unique index such that $x \in U_{i(x)}$. It follows that f is locally Lipschitz and $T \subseteq \partial^H f$ (resp. $T \subseteq \partial^F f$). $\qquad \square$

4   Cyclic submonotonicity

In Theorem 4 we obtained a sufficient condition ensuring that an operator T is included in the subdifferential $\partial^C f$ of a (locally Lipschitz) function f. In this section we reinforce Definition 3 - by using a notion of approximate subdivisions of closed polygonal paths - to ensure the coincidence of T with the subdifferential $\partial^C f$. This leads to a notion of cyclic submonotonicity, which turns out (in Section 5) to be a necessary and sufficient condition for the integration process described in Theorem 4. Its relation with radial cyclic submonotonicity is given in Proposition 17.

Let us first give the definition of a δ-subdivision of a closed polygonal path.

Definition 5 *Given $\delta > 0$ and a closed polygonal path $[w_h]_{h=1}^m$, we say that $\{x_i\}_{i=1}^n$ is a δ-subdivision of $[w_h]_{h=1}^m$, if $x_n = x_1$ and*
(i) $\{x_i\}_{i=1}^n \subseteq B_\delta([w_h]_{h=1}^m)$
(ii) $\|x_{i+1} - x_i\| < \delta$, for $i \in \mathbb{N}_{n-1}$
(iii) there exists a finite sequence $\{k_h\}_{h=1}^m$ with $1 = k_1 < k_2 < ... < k_m := n$ such that for $1 \leq h \leq m - 1$ we have:

$$k_h \leq i < k_{h+1} \implies \left\| \frac{x_{i+1} - x_i}{\|x_{i+1} - x_i\|} - \frac{w_{h+1} - w_h}{\|w_{h+1} - x_h\|} \right\| < \delta.$$

We are now ready to give the following definition.

Definition 6 *An operator T is called cyclically submonotone, if for any closed polygonal path $[w_h]_{h=1}^m$ and any $\varepsilon > 0$, there exists $\delta > 0$, such that for all δ-subdivisions $\{x_i\}_{i=1}^n$ of $[w_h]_{h=1}^m$ and all $x_i^* \in T(x_i)$ one has*

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \leq \varepsilon \sum_{i=1}^{n-1} \|x_{i+1} - x_i\|. \tag{21}$$

If U is an open subset of X, an operator T is said to be *cyclically submonotone on U* if (21) holds for closed polygonal paths and δ-subdivisions in U. Furthermore, a cyclically submonotone operator T on U is called *maximal cyclically submonotone* on U, if there is no cyclically submonotone operator $S \neq T$ such that $T(x) \subseteq S(x)$ for all $x \in U$.

Let us note that, as follows from (12), the length of a subdivision of a path is always equal to the length of the initial path. On the contrary, the definition of a δ-subdivision is more general, since no direct constraint on its length is imposed. This flexibility in Definition 5 enables us to show that - unlike the case of radial cyclic submonotonicity - every cyclically submonotone operator is also ds-submonotone.

Proposition 7 *Every cyclically submonotone operator on U is ds-submonotone on U. Consequently, if $U \subseteq \operatorname{int} \operatorname{dom}(T)$ then T is also locally bounded on U.*

Proof Let $x_0 \in U$, $\varepsilon > 0$ and $e \in S_X$. Since U is open, there exists $\lambda > 0$ such that $[x_0, x_0 + \lambda e] \subseteq U$. Let $m = 3$, $w_1 = x_0 = w_3$ and $w_2 = x_0 + \lambda e$. For $\varepsilon > 0$ and for the path $[w_h]_{h=1}^3$ take $\delta > 0$ as in Definition 6, and set $\delta' = \delta/2$. Then if $x_1, x_2 \in B_{\delta'}(x_0)$ are such that $x_1 \neq x_2$ and $\|\frac{x_1 - x_2}{\|x_1 - x_2\|} - e\| < \delta'$, we can easily check that for $x_3 = x_1$, $\{x_i\}_{i=1}^3$ is a δ-subdivision of the path $[w_h]_{h=1}^3$. So relation (21) yields (11) and T is ds-submonotone at x_0. Since x_0 is arbitrary in U, it follows that T is ds-submonotone on U. The last assertion follows from [11, Theorem 2.4]. $\qquad\square$

Remarks

1. Every cyclically submonotone operator is radially cyclically submonotone on every open subset U of its domain. Indeed, if (21) is true and $\{x_i\}_{i=1}^n$ is a subdivision of $[w_h]_{h=1}^m$ in U, then $\sum_{i=1}^{n-1} \|x_{i+1} - x_i\| = \sum_{i=1}^{n-1} \|w_{i+1} - w_i\|$. It follows that (21) yields (13) for $\varepsilon' = \varepsilon (\sum_{i=1}^{n-1} \|w_{i+1} - w_i\|)^{-1}$.

2. It is obvious that every cyclically monotone operator is cyclically submonotone. On the other hand, an operator can even be strongly monotone, without being cyclically submonotone, as illustrates the example (also used in [9] for a similar purpose) of the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$, with

$$T(x, y) = \langle \frac{x}{2} - y, x + \frac{y}{2} \rangle.$$

3. Cyclic submonotonicity is a separably determined property, i.e. an operator $T : X \rightrightarrows X^*$ is cyclically submonotone on U if, and only if, for every separable closed subspace Y of X, the operator $T \mid_Y : Y \rightrightarrows Y^*$ (defined for all $y_1, y_2 \in Y$ by $\langle T \mid_Y (y_1), y_2 \rangle := \langle T(y_1), y_2 \rangle$) is cyclically submonotone on $U \cap Y$.

We now show that if T is locally bounded and cyclically submonotone (resp. ds-submonotone) on U, then so does the w*-cusco generated by T. Let us recall that if U is an open subset of $\mathrm{dom}(T)$, the local boundedness assumption on T becomes superfluous, since it follows from its ds-submonotonicity (see Proposition 7).

Proposition 8 *If T is locally bounded and cyclically submonotone (resp. ds-submonotone) on U, then the operators $\overline{T}$ and $\overline{co}^{w^*}(T)$ are also cyclically submonotone (resp. ds-submonotone) on U. In particular, the w*-cusco generated by T is cyclically submonotone (resp. ds-submonotone) on U.*

Proof It is easily seen from (21) that $\overline{co}^{w^*}(T)$ is cyclically submonotone. In order to prove that $\overline{T}$ is cyclically submonotone, let us consider a closed polygonal path $[w_h]_{h=1}^m \subseteq U$ and $\varepsilon > 0$. Let us take $\delta > 0$ guaranteed by Definition 6 for the operator T and let us consider any δ-subdivision $\{x_i\}_{i=1}^n$ in U and $x_i^* \in \overline{T}(x_i)$. Then there exist nets $(x_i(\lambda))_{\lambda \in \Lambda}$ and $(x_i^*(\lambda))_{\lambda \in \Lambda}$ such that $x_i^*(\lambda) \in T(x_i(\lambda))$, $(x_i(\lambda)) \xrightarrow{\|\cdot\|} x_i$ and $(x_i^*(\lambda)) \xrightarrow{w^*} x_i^*$. Since $\{x_i\}_{i=1}^n$ is a finite sequence, there exists $\lambda_0 \in \Lambda$ such that $\{x_i(\lambda)\}_{i=1}^n$ is a δ-subdivision for all $\lambda \succeq \lambda_0$ (where $\succeq$ is the preorder relation of Λ). It follows from (21) that

$$\sum_{i=1}^n \langle x_i^*(\lambda), x_{i+1}(\lambda) - x_i(\lambda) \rangle \leq \varepsilon \sum_{i=1}^n \|x_{i+1}(\lambda) - x_i(\lambda)\|.$$

Taking limits in both sides we obtain

$$\sum_{i=1}^n \langle x_i^*, x_{i+1} - x_i \rangle \leq \varepsilon \sum_{i=1}^n \|x_{i+1} - x_i\|.$$

This shows that $\overline{T}$ is cyclically submonotone, hence so does the operator $\overline{co}^{w^*}(\overline{T})$. Since now T is locally bounded on U, we have

$$\overline{T}(x) := \bigcap_{\varepsilon > 0} \overline{\{T(x') : x' \in B_\varepsilon(x) \cap \mathrm{dom}(T)\}}^{w^*}$$

for all $x \in U$. Using Proposition 1 we conclude that $\overline{co}^{w^*}(\overline{T})$ is the minimal w*-cusco operator containing T. This finishes the proof. The assertions concerning ds-submonotonicity can be proved likewise. $\square$

The following proposition reveals an important feature of cyclic submonotonicity.

Proposition 9 *Let U be an open subset of $\mathrm{dom}(T)$ and T be a cyclically submonotone operator on U. The following statements are equivalent:*

(i) T is w*-cusco on U.

(ii) T is minimal w*-cusco on U.

(iii) T is maximal ds-submonotone on U.

(iv) T is maximal cyclically submonotone on U.

Proof Implication $(ii) \Rightarrow (i)$ is obvious. Assume now that (i) holds. Using Proposition 7 we conclude that T is ds-submonotone on U. Since T is w*-cusco, from [11, Lemma 3.2] it follows that T is maximal submonotone on U. Hence $(i) \Rightarrow (iii)$.

$(iii) \Rightarrow (iv)$: Let S be a cyclically submonotone operator whose graph contains the graph of T. Then S is ds-submonotone (see Proposition 7), hence it coincides with T.

$(iv) \Rightarrow (ii)$: Since T is locally bounded on $U \subseteq \mathrm{intdom}(T)$, Proposition 8 guarantees that T is w*-cusco. Assume that there exists $S \subseteq T$ such that S is w*-cusco. Obviously S will also be cyclically submonotone. Since $(i) \Rightarrow (iv)$, S is maximal cyclically submonotone, whence $S = T$. $\square$

Remark 10 *We recall from [23] that if X is Asplund (resp. X has a Gâteaux differentiable norm), then every minimal w*-cusco operator is single-valued and $(\|.\| - \|.\|)$ upper semicontinuous (resp. $(\|.\| - w^*)$ upper semicontinuous) at every point of a G_δ dense set.*

Corollary 11 *Let T be a cyclically submonotone operator on an open subset U. Then $\overline{co}^{w^*}(T)$ is maximal cyclically submonotone on U.*

Proof Since T is locally bounded, the operator $S := \overline{co}^{w^*}(\overline{T})$ is w*-cusco. By Proposition 8, S is also cyclically monotone. The conclusion follows from Proposition 9. $\square$

Let further Z be a closed subspace of X and define the multivalued operator $S : Z \rightrightarrows Z^*$ as follows

$$S(z) = \{z^* \in Z^* : \exists x^* \in T(z) \text{ such that } z^* = x^* \mid_Z\} \tag{22}$$

(where $x^* \mid_Z$ denotes the restriction of the functional x^* to Z).

Lemma 12 *(i) If T is locally bounded and w*-cusco on an open subset U, then S is $\sigma(Z^*, Z)$-cusco on $U \cap Z$, where $\sigma(Z^*, Z)$ denotes the w*-topology on Z^*.*

(ii) If T is maximal cyclically submonotone on U, then S is maximal cyclically submonotone on $U \cap Z$.

Proof Assertion (i) follows easily since S has a $\sigma(Z^*, Z)$-closed graph on $U \cap Z$. To show (ii), let us observe (from Definition 6) that if T is cyclically submonotone on U, then S is cyclically submonotone on $U \cap Z$. By Propositions 7 and 9 we have that T is locally bounded and w*-cusco. It follows by (i) that S is $\sigma(Z^*, Z)$-cusco on $U \cap Z$, so the proof finishes by a new application of Proposition $9(i) \rightarrow (iv)$. $\square$

5 Main results

Throughout this section U will always denote a non-empty open subset of X. Let us give the following definition.

Definition 13 *A (locally Lipschitz) function $f : U \to \mathbb{R}$ is called subsmooth if $\partial^C f$ is ds-submonotone on U.*

Every subsmooth function f is regular (see [24], [11, Theorem 4.1]), a locally Lipschitz function f being called *regular* ([6]) if $f'(x, d) = f^o(x, d)$, for all $d \in X$. It follows that

$$\partial^C f = \partial^H f. \tag{23}$$

In the particular case where X has some regularity, subsmoothness is characterized as follows.

Proposition 14 *Let X be an Asplund space (resp. X has a Gâteaux differentiable renorming). Then a locally Lipschitz function $f : U \to \mathbb{R}$ is subsmooth if, and only if, $\partial^F f$ (resp. $\partial^H f$) is ds-submonotone on U.*

Proof The "necessity" part is obvious. To show the "sufficiency" part, let $T = \partial^F f$ (resp. $\partial^H f$). Since f is locally Lipschitz and T is included in $\partial^C f$, it follows that T is locally bounded. Using Corollary 2 and Proposition 8 we obtain that $\partial^C f$ is ds-submonotone, hence that f is subsmooth. (Note that this implies that $\text{dom}(\partial^H f) = U$). $\qquad \square$

It follows from Definition 13 and the comments after the definition of ds-submonotonicity (in Section 2) that every convex or continuously differentiable (i.e. C^1) function is subsmooth. One of the main results in [24] is the following characterization of subsmooth functions in finite dimensions: a (locally Lipschitz) function f on $U \subseteq \mathbb{R}^n$ is *subsmooth* if, and only if, it is lower C^1, a function f being called *lower C^1* if for each $x_0 \in U$, there exist a neighborhood V of x_0, a compact set S and a jointly continuous function $g : V \times S \to \mathbb{R}$ such that for all $x \in V$, $f(x) = \max_{s \in S} g(x, s)$ and $D_x g$ (exists and) is jointly continuous. In the last section we give some typical examples of subsmooth functions in infinite dimensions.

We now state the main results of the paper.

THEOREM A *For a locally Lipschitz function $f : U \to \mathbb{R}$ the following are equivalent:*
(i) *f is subsmooth*
(ii) *$\partial^C f$ is maximal cyclically submonotone on U.*

THEOREM B *Let U be an open connected subset of X and f_1, f_2 be two subsmooth (or more generally regular) functions on U such that $\partial^C f_1 = \partial^C f_2$. Then $f_1 = f_2 + c$ for some $c \in \mathbb{R}$.*

THEOREM C *If $T : X \rightrightarrows X^*$ is a multivalued operator and U an open subset of $\text{dom}(T)$, then T is maximal cyclically submonotone on U if, and only if, $T = \partial^C f = \partial^H f$*

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for some subsmooth function $f : U \to \mathbb{R}$, which is unique (up to a constant) on every connected subset of U. If, in addition, T is submonotone, then $T = \partial^F f$.

Proof of Theorem A The implication $(ii) \Rightarrow (i)$ is clear in view of Proposition 7 and Definition 13.

For the implication $(i) \Rightarrow (ii)$, set $T := \partial^C f$. Since T is w*-cusco, in view of Proposition 9 $(i) \Rightarrow (iv)$, it clearly suffices to show that T is cyclically submonotone. To this end, consider any closed polygonal path $[w_h]_{h=1}^m \subseteq U$ and any $\varepsilon > 0$. Set $C = [w_h]_{h=1}^m$ and

$$e_h = \frac{w_{h+1} - w_h}{\|w_{h+1} - w_h\|}$$

for $h \in \mathbb{N}_{m-1} := \{1, 2, ..., m-1\}$.

Since T is ds-submonotone, it follows that for every $x \in C$ and $h \in \mathbb{N}_{m-1}$ there exists $\alpha(x, h) > 0$ such that

$$\frac{\langle x_1^* - x_2^*, x_2 - x_1 \rangle}{\|x_1 - x_2\|} < \varepsilon \tag{24}$$

whenever $x_1 \neq x_2$ with $\|x_i - x\| < \alpha(x, h)$, $x_i^* \in T(x_i)$ $(i = 1, 2)$ and

$$\|\frac{x_1 - x_2}{\|x_1 - x_2\|} - e_h\| < \alpha(x, h).$$

Set $\beta(x) := \min_{h \in \mathbb{N}_m} \alpha(x, h)$ and note that (24) holds for all $x_1 \neq x_2$ such that $x_i \in B_{\beta(x)}(x)$ $(i = 1, 2)$ and $\frac{x_1 - x_2}{\|x_1 - x_2\|} \in \bigcup_{h \in \mathbb{N}_{m-1}} B_{\beta(x)}(e_h)$.

Let $\delta > 0$ be a Lebesgue number of the open covering $\left(B_{\beta(x)}(x)\right)_{x \in C}$ of the compact set C, i.e.

$$\forall w \in C, \exists x \in C : B_\delta(w) \subseteq B_{\beta(x)}(x). \tag{25}$$

Let us consider any δ-subdivision $\{x_i\}_{i=1}^n$ of $C = [w_h]_{h=1}^m$. Since f is locally Lipschitz, using Lebourg's Mean Value theorem ([13]) on every segment $[x_i, x_{i+1}]$ (for $i \in \mathbb{N}_{n-1}$), we infer the existence of $z_i \in]x_i, x_{i+1}[$ and $z_i^* \in T(z_i)$ such that

$$f(x_{i+1}) - f(x_i) = \langle z_i^*, x_{i+1} - x_i \rangle. \tag{26}$$

Adding the above equalities, we have

$$\sum_{i=1}^{n-1} \langle z_i^*, x_{i+1} - x_i \rangle = 0$$

which yields

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle = \sum_{i=1}^{n-1} \langle x_i^* - z_i^*, x_{i+1} - x_i \rangle. \tag{27}$$

Since $\{x_i\}_{i=1}^n$ is a δ-subdivision of C, it follows from (24) and (25) that for every $i \in \mathbb{N}_{n-1}$

$$\langle x_i^* - z_i^*, \frac{x_{i+1} - x_i}{\|x_{i+1} - x_i\|} \rangle = \langle x_i^* - z_i^*, \frac{z_i - x_i}{\|z_i - x_i\|} \rangle < \varepsilon$$

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which, combined with (27), yields

$$\sum_{i=1}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle = \varepsilon \sum_{i=1}^{n-1} \|x_{i+1} - x_i\|.$$

This finishes the proof. $\qquad\square$

Before proceeding to the proof of Theorem B, we shall need the following easy result.

Lemma 15 *Let $f : U \to \mathbb{R}$ be a locally Lipschitz function, let Z be a closed linear subspace of X and consider the function $g : Z \cap U \to \mathbb{R}$ defined by*

$$g(z) = f(z) \text{ for all } z \in U \cap Z. \tag{28}$$

If $S : Z \rightrightarrows Z^*$ is as in (22) for $T = \partial^H f$, then we have:
  (i) $S(z) \subseteq \partial^H g(z)$ for all $z \in U \cap Z$.
  (ii) If for some $x_0 \in U \cap Z$, $\partial^H f(x_0) = \partial^C f(x_0)$, then $S(x_0) = \partial^H g(x_0) = \partial^C g(x_0)$.

Proof (i) Let $z \in U \cap Z$ and $z^* \in S(z)$. Then $z^* = x^* \mid_Z$ for some $x^* \in \partial^H f(z)$. Since f is locally Lipschitz, by (5) we conclude that $g'(z, u) = f'(z; u) \geq \langle x^*, u \rangle$, for all $u \in Z$. It follows that $z^* \in \partial^H g(z)$.

(ii) Suppose now that for some $x_0 \in U \cap Z$, $\partial^H f(x_0) = \partial^C f(x_0)$. By (i) we have $S(x_0) \subseteq \partial^H g(x_0) \subseteq \partial^C g(x_0)$. Let us show that $\partial^C g(x_0) \subseteq S(x_0)$. Indeed, let $z_0^* \in \partial^C g(x_0)$. Then we have from (4) that

$$\langle z_0^*, u \rangle \leq g^o(x_0; u) \leq f^o(x_0; u), \text{ for all } u \in Z.$$

Using the Hahn Banach theorem, we conclude the existence of some $x^* \in X^*$ such that $x^* \mid_Z = z^*$ and $\langle x^*, . \rangle \leq f^o(x_0; .)$, so that $x^* \in \partial^C f(x_0) = \partial^H f(x_0)$. It follows that $z^* \in S(x_0)$. $\qquad\square$

Proof of Theorem B Suppose that f_1, f_2 are two subsmooth functions on U such that $\partial^C f_1 = \partial^C f_2$. Without loss of generality we suppose that $0 \in U$. For any $x \in X$ let us set $Z = \text{span}[x]$, $g_i = f_i \mid_Z$ (the restriction of f_i to Z, $i = 1, 2$) and $T = \partial^H f_1$. By (23) we have

$$T = \partial^H f_1 = \partial^C f_1 = \partial^C f_2 = \partial^H f_2$$

and by Lemma 15(ii)

$$\partial^H g_1 = \partial^C g_1 = \partial^C g_2 = \partial^H g_2. \tag{29}$$

Since Z is a one-dimensional space (in fact separable would suffice), it follows from [3, Theorem 5.12] (see also [5, Section 4.2]) that the regular functions g_1, g_2 are essentially smooth.

Case 1: Suppose that U is convex.

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Then the set $U \cap Z$ is connected, so relation (29) yields $g_1 = g_2 + c$ for some $c \in \mathbb{R}$ (see [5, Proposition 4.12] or [3, Proposition 5.9]). Since $g_i = f_i \mid_Z$, we obtain

$$f_1(x) - f_2(x) = c = f_1(0) - f_2(0).$$

Since x is arbitrarily chosen, we obtain $f_1 = f_2 + c$ on U.

Case 2: (general) Since U is open and X is locally convex, from case 1 we conclude that $f_1 - f_2$ is locally constant on U. Since now U is connected, it follows that $f_1 - f_2$ is constant on U. The proof is complete. □

Let us now proceed to the proof of Theorem C. We shall need the following lemma.

Lemma 16 *Let $f : U \to \mathbb{R}$ be locally Lipschitz, $x_0 \in X$ and Y be a separable subspace of X. Then there exists a separable subspace Z of X containing Y and x_0 such that for the function $g : Z \cap U \to \mathbb{R}$ given by $g = f \mid_{U \cap Z}$ (as in (28)) we have*

$$g^o(x_0; u) = f^o(x_0; u) \text{ for all } u \in Z \tag{30}$$

and consequently

$$\partial^C g(x_0) = \{z^* \in Z^* : z^* = x^* \mid_Z, z^* \in \partial^C f(x_0)\}. \tag{31}$$

Proof Let $Y_0 = \text{span}[Y, x_0]$ be the closed linear space generated by Y and $\{x_0\}$ and let D_0 be a countable dense subset of Y_0. Then for every $d \in D_0$, there exist $\{x_n\}_{n \geq 1}$ in X and $\{t_n\}_{n \geq 1}$ in $]0, 1[$ such that $(x_n) \longrightarrow x_0$, $(t_n) \to 0^+$ and

$$f^o(x_0; d) < \frac{f(x_n + t_n d) - f(x_n)}{t_n} + \frac{1}{n}. \tag{32}$$

Set $A_0(d) = \{x_n : n \in \mathbb{N}\}$ and $A_0 = \bigcup_{d \in D_0} A_0(d)$. Consider the separable space $Y_1 = \text{span}[Y_0, D_0]$, let D_1 be a countable dense subset of Y_1 and define (using (32)) $A_1(d)$ for all d in D_1 as above and $A_1 = \bigcup_{d \in D_1} A(d)$. Proceeding like this, we obtain an increasing sequence of closed separable subspaces Y_n of X and a sequence (D_n) of countable subsets such that D_n is dense in Y_n. Set

$$Z = \bigcup_n Y_n$$

and $g = f \mid_Z$. Then for any $u \in Z$ and $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and $d \in D_n$ such that $\|u - d\| < \varepsilon$. Using (32) we conclude easily that $g^o(x_0; d) = f^o(x_0; d)$. Since the functions $u \mapsto g^o(x_0; u)$ and $u \mapsto f^o(x_0; u)$ are Lipschitz, (30) follows. Relation (31) is now an easy consequence of (30) and of the Hahn-Banach theorem. □

Proof of Theorem C The sufficiency part following from Theorem A, we only have to show the necessity part.

To this end, let us suppose that T is maximal cyclically submonotone on U. In particular, T is a locally bounded ds-submonotone and radially cyclically submonotone

operator on U (see Proposition 7 and Remark 1). It follows by Theorem 4 that for all $x \in U$, $T(x) \subseteq \partial^H f(x)$, where f is given by (17). Let us show that $T(x) = \partial^C f(x)$ for all $x \in U$.

Suppose that the contrary holds. Then for some $x_0 \in X$ and $x_0^* \in \partial f^C(x_0)$ we have $x_0^* \notin T(x_0)$. By Corollary 9, $T(x_0)$ is a non-empty w*-closed convex set, hence there exists $u \in X$ such that

$$\langle x_0^*, u \rangle > \sup_{x^* \in T(x_0)} \langle x^*, u \rangle. \tag{33}$$

Set $Y = \mathrm{span}[x_0, u]$, and consider the separable subspace Z of X given by Lemma 16 and $g = f \mid_Z$. Let $S : Z \rightrightarrows Z^*$ be as in (22), i.e. for every $z \in U \cap Z$

$$S(z) = \{z^* \in Z^* : z^* = x^* \mid_Z \text{ for some } x^* \in T(z)\}.$$

Then by Lemma 12 S is $\sigma(Z^*, Z)$-cusco on $U \cap Z$ and by Lemma 15(i)

$$S \subseteq \partial^H g.$$

Since S has non-empty values on $U \cap Z$, the above relation yields that $S(x) = \{D^H g(x)\}$ for all points x for which the Hadamard derivative $D^H g(x)$ exists. Since Z is a separable Banach space, it admits a Gâteaux smooth renorming. It follows that the Clarke subdifferential $\partial^C g$ is given by (7) and is the smallest $\sigma(Z^*, Z)$-cusco mapping whose graph contains the graph of the Hadamard derivative $D^H g$. Since S is $\sigma(Z^*, Z)$-cusco, it follows that $\mathrm{Gr}(\partial^C g) \subseteq \mathrm{Gr}(S)$, for all $x \in U$. Since $\mathrm{Gr}(\partial^H g) \subseteq \mathrm{Gr}(\partial^C g)$, we conclude that

$$S = \partial^H g = \partial^C g \text{ on } U \cap Z.$$

In particular $S(x_0) = \partial^C g(x_0)$, and using the conclusion of Lemma 16

$$S(x_0) = \{z^* \in Z^* : z^* = x^* \mid_Z, \ x^* \in \partial^C f(x_0)\}.$$

Let $z_0^* := x_0^* \mid_Z \in S(x_0) = \partial^C g(x_0)$. Since $u \in Z$, it follows that

$$\langle x_0^*, u \rangle = \langle z_0^*, u \rangle \leq g^o(x_0; u) = f^o(x_0; u).$$

This yields a contradiction to (33) since

$$g^o(x_0; u) = \sup_{z^* \in \partial^C g(x_0)} \langle z^*, u \rangle = \sup_{x^* \in S(x_0)} \langle x^*, u \rangle = \sup_{x^* \in T(x_0)} \langle x^*, u \rangle.$$

Hence we have shown that $T = \partial^C f$ on U. It follows from Definition 13 that f is subsmooth and by Theorem B that it is unique (modulo a constant) in every connected subset of U.

If moreover T is submonotone, then using again Theorem 4 we infer that $T(x) \subseteq \partial^F f(x)$, for all $x \in U$. Hence $\mathrm{Gr}(T) \subseteq \mathrm{Gr}(\partial^F f) \subseteq \mathrm{Gr}(\partial^C f)$ and $T = \partial^C f$ (on U) whence $T = \partial^F f = \partial^C f$ on U. $\qquad\square$

An inspection of the above proof yields the following result.

Proposition 17 *Suppose that T is a locally bounded w*-cusco operator on an open subset U of X. Then T is (maximal) cyclically submonotone on U if, and only if, T is radially cyclically submonotone and ds-submonotone on U.*

Proof If T is cyclically submonotone, then from Proposition 7 and Remark 1 it follows that T is radially cyclically submonotone and ds-submonotone on U. Conversely, if T is radially cyclically submonotone and ds-submonotone on U, then by Theorem 4 we infer that $T \subseteq \partial^H f$ on U for some locally Lipschitz function f.

Since now T is locally bounded and w*-cusco on U, by Lemma 12(i), for every closed subspace Z of X, the operator S given in (22) is $\sigma(Z^*, Z)$-cusco on $U \cap Z$. Thus, repeating the arguments of the above proof, we obtain that $T = \partial^C f$ on U and that f is subsmooth. It follows from Theorem A that T is (maximal) cyclically submonotone on U. $\qquad\square$

6 Examples of subsmooth functions

Apart from the classes of convex continuous or C^1-functions (or of sums of such functions), typical examples of subsmooth functions are also certain types of marginal functions, as for instance the class of lower C^1 functions introduced in [24] (and also considered in [21] and [16]). Let us note that subdifferentiability properties of marginal functions have been studied by many authors, see for instance [3], [7], [8] and [11].

In the sequel let A be an arbitrary non-empty set and U an open subset of X. We consider the marginal function $f : U \to \mathbb{R}$ defined for every $x \in U$ by

$$f(x) = \sup_{\alpha \in A} g(x, \alpha), \tag{34}$$

where $g : U \times A \to \mathbb{R}$ is such that $g(\cdot, \alpha)$ is a regular locally Lipschitz function and $f(x) < +\infty$ for every $x \in U$. Let us further assume that

(i) for every $x_0 \in U$, there exists $\delta > 0$ such that the set

$$\bigcup \{\partial^C g(x, \alpha) : \alpha \in A, x \in B_\delta(x_0), g(x, \alpha) \geq f(x_0) - \delta\}$$

is norm bounded.

(ii) for every $x \in X$ and $e \in S_X$ there exists $\varepsilon > 0$ such that for every $\gamma > 0$ there exists $\delta > 0$ such that

$$f'(y, a; e) - f'(x, a; e) < \gamma$$

whenever $\| x - y \| < \delta$ and $g(x, \alpha) \geq f(x) - \varepsilon$.

(iii) for every $x_0 \in U$, $e \in S_X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$g'(x, a; e) - \frac{g(x + te, a) - g(x, a)}{t} < \varepsilon \tag{35}$$

for all $x \in B_\delta(x_0)$, all $t \in]0, \ \delta[$ and all $a \in A$ with $g(x, a) \geq f(x_0) - \delta$.

18

The following result is an easy consequence of results established in [11].

Theorem 18 *If f is a marginal function (relation (34)) satisfying conditions (i)-(iii), then f is subsmooth on U and $\partial^H f = \partial^C f$ is maximal cyclically submonotone on U.*

Proof By [11, Theorem 5.4 (a)], f is locally Lipschitz, and by [11, Lemma 5.3] and [11, Theorem 5.2 (d)], f is regular, therefore $\partial^H f = \partial^C f$. Now again by [11, Theorem 5.4 (a)], $\partial^C f$ is ds-submonotone in U, i.e. f is subsmooth. Theorem A finishes the proof. $\square$

Remarks

1. It is easily seen that the above class of functions contains the class of lower C^1 functions. Combining this with Theorem 18 and Spingarn's characterization of lower C^1 functions ([24, Theorem 3.9]) we conclude that in finite dimensions a function f is lower C^1 if, and only if, f is given by (34) and satisfies conditions (i)-(iii).

2. If in addition to the assumptions of Theorem 18 the choice of δ in (35) does not depend on e, then by [11, Theorem 5.4 (b)] we conclude that $\partial^C f$ is submonotone, and by Theorem C, that $\partial^F f = \partial^C f$.

Let now A be an arbitrary nonempty subset of the Banach space X. Let us define the *distance function* by

$$d_A(x) := \inf_{\alpha \in A} \| x - \alpha \| \qquad (x \in X).$$

The following proposition provides another typical example of subsmooth function.

Proposition 19 *Suppose that the norm of X is uniformly Gâteaux (resp. uniformly Fréchet) differentiable. For any nonempty closed set A of X let us consider the function*

$$f(x) = -d_A(x).$$

(i) Then f is subsmooth, hence regular, on $X \setminus A$.
(ii) $\partial^C f = \partial^H f$ (resp. $\partial^C f = \partial^F f$) is maximal cyclically submonotone and $\partial^C d_A(.)$ is minimal w*-cusco on $X \setminus A$.

Proof The assertions follow from [11, Th. 5.6 (a),(b)], Theorem A and Remark 2. $\square$

Let us now consider another important class of examples of subsmooth functions. We shall say that a function $f : U \to \mathbb{R}$ is *amenable* ([22, Definition 10.23]) if for any $x_0 \in U$ there exist an open neighborhood V of x_0, a Banach space Y, a continuously differentiable function $F : V \to Y$ and a proper lower semicontinuous convex function $g : Y \to \mathbb{R} \cup \{+\infty\}$ such that

$$f(x) = g(F(x)) \text{ for all } x \in V \tag{36}$$

and

$$\mathbb{R}_+ (\text{dom } g - F(x_0)) + F'(x_0)(X) = Y. \tag{37}$$

For the sake of simplicity, and since in this paper we limit our study to locally Lipschitz functions, we further consider the subclass $\mathcal{A}(U)$ of amenable functions f such that in the decomposition (36) we have $F(V) \subseteq \operatorname{int} \operatorname{dom}(g)$. Obviously every function in $\mathcal{A}(U)$ is locally Lipschitz and condition (37) is satisfied.

Proposition 20 *If $f \in \mathcal{A}(U)$, then $\partial^C f$ is s-submonotone (hence in particular f is subsmooth).*

Proof Let f be in $\mathcal{A}(U)$. With no loss of generality we may assume that $V = U$ so that $f = g \circ F$ with g and F as in (36). Since g is regular on $F(U)$ (as it is convex and continuous on $\operatorname{int} \operatorname{dom}(g)$), applying [6, Theorem 2.3.10] we conclude that f is also regular, that is $\partial^C f = \partial^H f$. Set now $T = \partial^C f = \partial^H f$. We shall show that T is s-submonotone.

To this end, let $x_0 \in U$, $x_1, x_2 \in U$ and $x_i^* \in \partial f(x_i)$, $i = 1, 2$. Then there exist $y_i^* \in \partial g(F(x_i))$ $(i = 1, 2)$ such that $x_i^* = y_i^* \circ F'(x_i)$, where $F'(x)$ denotes the Fréchet derivative of F at x. It follows that

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle = \langle y_1^*, F'(x_1)(x_1 - x_2) \rangle - \langle y_2^*, F'(x_2)(x_1 - x_2) \rangle. \tag{38}$$

Since F is continuously differentiable, there exist $\delta > 0$ and a function $r : U \times U \to Y$ such that

$$F(v) - F(u) = F'(u)(v - u) + r(u, v) \tag{39}$$

for all $u, v \in B_\delta(x_0)$ and

$$\lim_{\substack{u,v \to x_0 \\ u \neq v}} \frac{\|r(u, v)\|}{\|u - v\|} = 0. \tag{40}$$

Combining (38) with (39) we obtain thanks to the monotonicity of ∂g that

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq \langle y_1^*, r(x_1, x_2) \rangle + \langle y_2^*, r(x_2, x_1) \rangle$$

which yields, when $x_1 \neq x_2$, that

$$\frac{\langle x_1^* - x_2^*, x_1 - x_2 \rangle}{\|x_1 - x_2\|} \geq \langle y_1^*, \frac{r(x_1, x_2)}{\|x_1 - x_2\|} \rangle + \langle y_2^*, \frac{r(x_2, x_1)}{\|x_1 - x_2\|} \rangle.$$

The result now follows from (40) and the local boundedness of ∂g near $F(x_0)$. $\qquad\square$

Remark Since every strictly Gâteaux differentiable function $F : U \to Y$ is locally Lipschitz ([6, Proposition 2.2.1]), a slight modification of the above proof suffices to establish that $\partial^C f$ is ds-submonotone on U, whenever F is strictly Gâteaux differentiable and g is locally Lipschitz with $\partial^C g$ s-submonotone on an open set containing $F(U)$.

References

[1] BIRGE, R. & QI, L., Semi-regularity and generalized subdifferentials with applications to optimization, *Math. Oper. Res.* **18** (1993), 982-1005.

[2] BORWEIN, J.M., Minimal cuscos and subgradients of Lipschitz functions, in: *Fixed Point Theory and its Applications*, (J.-B. Baillon & M. Théra eds.), Pitman Res. Notes in Math. Series, No. 252, Longman, Essex, (1991), 57-82.

[3] BORWEIN, J. & MOORS, W., Essentially smooth Lipschitz functions, *J. Funct. Anal.* **149** (1997), 305-351.

[4] BORWEIN, J. MOORS, W. & SHAO, Y., Subgradient representation of multifunctions, *J. Austral. Math. Soc. (Series B)* **40** (1998), 1-13.

[5] BORWEIN, J. & ZHU, Q., Multivalued and functional analytic techniques in nonsmooth analysis, (F.H. Clarke & R. J. Stern eds.), *Nonlinear Analysis, Differential Equations and Control* (1999), 61-157.

[6] CLARKE, F.H., *Optimization and Nonsmooth Analysis*, Wiley Interscience, New York, New York (1983).

[7] CORREA, R. & JOFRE, A., Tangentially continuous directional derivatives in nonsmooth analysis, *J. Opt. Th. Appl.* **61** (1989), 1-21.

[8] CORREA, R. & THIBAULT, L., Subdifferential analysis of bivariate separately regular functions, *J. Math. Anal. Appl.* **148** (1990), 157-174.

[9] DANIILIDIS, A. & HADJISAVVAS, N., On the subdifferentials of quasiconvex and pseudoconvex functions and cyclic monotonicity, *J. Math. Anal. Appl.* **237** (1999), 30-42.

[10] GEORGIEV, P., Submonotone mappings in Banach spaces and differentiability of nonconvex functions, *Compt. Rend. Acad. Bulg. Sci.* **42** (1989), 13-16.

[11] GEORGIEV, P., Submonotone mappings in Banach spaces and applications, *Set-Valued Analysis* **5** (1997), 1-35.

[12] JANIN, R., Sur des multiapplications qui sont des gradients généralisés, *C.R. Acad. Sc. Paris* **294** (1982), 117-119.

[13] LEBOURG, G., Generic differentiability of Lipschitzian functions, *Trans. Amer. Math. Soc.* **256** (1979), 125-144.

[14] MIFFLIN, R., Semismooth and semiconvex functions in constrained optimization, *SIAM J. Control Optim.* **15** (1977), 959-972.

[15] MOORS, W., A Characterization of minimal subdifferential mappings of locally Lipschitz functions, *Set-Valued Analysis* **3** (1995), 129-141.

[16] PENOT, J.-P., Favorable classes of mappings and multimappings in nonlinear analysis and optimization, *J. Convex Analysis* **3** (1996), 97-116.

[17] POLIQUIN, R., Integration of subdifferentials of nonconvex functions, *Nonlinear Analysis TMA* **17** (1991), 385-398.

[18] PREISS, D., Differentiability of Lipschitz functions on Banach spaces, *J. Functional Analysis* **91** (1990), 312-345.

[19] QI, L., The maximal normal operator space and integration of subdifferentials of nonconvex functions, *Nonlinear Analysis TMA* **13** (1989), 1003-1011.

[20] ROCKAFELLAR, R.T., On the maximal monotonicity of subdifferential mappings, *Pacific J. Math.* **33** (1970), 209-216.

[21] ROCKAFELLAR, R.T., 'Favorable classes of Lipschitz continuous functions in subgradient optimization' in *Nondifferentiable Optimization* (1982), Nurminski E. (eds), Pergamon Press, New York.

[22] ROCKAFELLAR, R.T. & WETS, J.-B., *Variational Analysis*, Springer, New York (1998).

[23] PREISS, D., PHELPS, R. & NAMIOKA, I., Smooth Banach spaces, weak Asplund spaces and monotone or USCO mappings, *Israel J. Math.* **72** (1990), 257-279.

[24] SPINGARN, J.E., Submonotone subdifferentials of Lipschitz functions, *Trans. Amer. Math. Soc.* **264** (1981), 77-89.

[25] THIBAULT, L. & ZAGRODNY, D., Integration of subdifferentials of lower semi-continuous functions on Banach spaces, *J. Math. Anal. Appl.* **189** (1995), 33-58.

[26] WANG, X., *Fine and pathological properties of subdifferentials*, Ph.D. Dissertation (1999), Simon Fraser University, Vancouver, Canada.

[27] WU, Z. & YE, J., Some results on integration of subdifferentials, *Nonlinear Analysis TMA* **39** (2000), 955-976.

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