HOROFUNCTION EXTENSION
AND METRIC COMPACTIFICATIONS

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Abstract. A necessary and sufficient condition for the horofunction extension $\overline{(X,d)}^h$ of a metric space (X,d) to be a compactification is hereby established. The condition clarifies previous results on proper metric spaces and geodesic spaces and yields the following characterization: a Banach space is *Gromov-compactifiable* under any renorming if and only if it does not contain an isomorphic copy of ℓ^1. In addition, it is shown that, up to an adequate renorming, every Banach space is Gromov-compactifiable. Therefore, the property of being Gromov-compactifiable is not invariant under bi-Lipschitz equivalence.

Key words: Metric space, normed space, compactification, horofunction extension, ℓ^1-criterium.

AMS Subject Classification: *Primary* 53C23, 54D35, 46B03 ; *Secondary* 46B20, 51F99.

Contents

1. Introduction and preliminary results

A *compactification* for a topological space X is a pair (Y, i), where Y is a compact space and $i : X \to Y$ is a continuous injection such that $i(X) \subset Y$ is dense and i is a homeomorphism from X to $i(X)$. If the injection i is canonical or implicitly known, we simply say that the compact space Y is a compactification of the space X.

Classical instances of compactification are the Alexandroff (one-point) compactification X_∞ (for locally compact spaces) and the Stone-Čech compactification βX (for completely regular spaces), corresponding to the two extreme cases in terms of size.

Gromov [15] proposed a new compactification scheme in case that (X, d) is a metric space. This is based on the identification of each point z of the space with the distance function $d(\cdot, z)$ to it (modulo constant functions), providing a natural injection of X to a quotient of the space of continuous real functions on X (endowed with the compact-open topology). Gromov called *horofunction extension* of X the closure $\overline{X}^h$ of the image of X there (see more details in Subsection 1.1). For applications of the horofunction extension and related constructions in more abstract settings, we refer to [1, 12, 25, 26].

In general, the horofunction extension of a metric space (X, d) is not a (topological) compactification of X, since the aforementioned injection of the space does not necessarily yield a homeomorphism over its image. In this work we use the following terminology:

Definition 1.1 (Gromov-compactification). *We say that a metric space (X, d) is Gromov-compactifiable if the horofunction extension $\overline{X}^h$ is a (topological) compactification for X.*

There are several known examples of Gromov-compactifiable spaces, as for instance proper geodesic spaces [3] or Hilbert spaces [24] and [17], as well as sufficient criteria on the space ensuring this property [11]. However, a complete characterization of Gromov-compactifiability was still up-to-date unavailable. The current work aims to fulfill this gap.

Our main contributions are:

- A necessary and sufficient condition for a metric space (X, d) to be Gromov-compactifiable (Theorem 2.1).
- (characterization for normed spaces) A normed space $(X, \|\cdot\|)$ is not Gromov-compactifiable *if and only if* the Hausdorff distance of the sphere of any finite dimensional subspace of X to the sphere of X is equal to 2 (Theorem 3.2 $(a) \Leftrightarrow (c)$).
- (ℓ^1-criterium) A normed space is Gromov-compactifiable under any renorming *if and only if* it does not contain an isomorphic copy of ℓ^1 (Theorem 3.7).
- Every Banach space can be renormed to become Gromov-compactifiable (Corollary 3.11). In particular, Gromov-compactifiability is not invariant under bi-Lipschitz homeomorphisms.

The paper is organized as follows: in the rest of this section we review the State-of-the-art and provide motivation for this study. Section 2 contains our main results in metric spaces. In Subsection 2.1 we establish a general characterization of Gromov-compactifiability of metric spaces. Some applications are also given for specific types of spaces, in particular, for locally compact or proper metric spaces. In Subsection 2.2 we compare the horofunction extension of a locally compact space with the so-called *metric compactification*, introduced by Rieffel in [23]. Section 3 is devoted to study Gromov-compactifiability in the setting of normed spaces. In Subsection 3.1 we obtain a geometric characterization, in terms of the Hausdorff distance between spheres. In Subsection 3.2 we obtain a connection with the so-called *octahedrality* of the norm, a property introduced in [13]. Furthermore, we characterize stability of Gromov-compactifiability under renormings, in terms of non-containment of an isomorphic copy of ℓ^1. In Subsection 3.3 we give a variety of examples and applications, in particular to Lipschitz-free spaces and 1-Wasserstein spaces.

1.1. **Original definition of the horofunction extension.** The notion of *horofunction extension* of a metric space (X, d) goes back to Gromov in [15] (see also [3]) and was defined as follows. For each $z \in X$, consider the distance function $d_z := d(\cdot, z)$. It is not difficult to see that the mapping

$$\iota : X \to C(X)$$

defined by $\iota(z) := d_z$ is a topological embedding of X into the space $C(X)$ of continuous real functions on X, endowed with the compact-open topology. We introduce the equivalence relation in $C(X)$ given by $f \sim g$ if, and only if, $f - g$ is constant. Denote by $\widehat{C}(X)$ the corresponding quotient space and by $\pi : C(X) \to \widehat{C}(X)$ the natural quotient map. It is easy to check that $\hat{\iota} := \pi \circ \iota : X \to \widehat{C}(X)$ is one-to-one. Therefore, we have that $\hat{\iota} : X \to \widehat{C}(X)$ is a *continuous injection*. Now we define the *horofunction extension* $\overline{X}^h$ of X as the closure of $\hat{\iota}(X)$ in $\widehat{C}(X)$, and we call $\overline{X}^h \setminus X$ the *horofunction boundary* of X. The elements of $\overline{X}^h \setminus X$ will be called *horofunctions* on X. In the literature, the elements of $\overline{X}^h$ are also called *metric functionals* on X (see e.g.[20] and references therein.)

On the other hand, if we fix an arbitrary point $x_0 \in X$ and we consider the closed subspace $C_{x_0}(X)$ of $C(X)$ formed by all continuous real functions on X vanishing at x_0, we see that $\widehat{C}(X)$ is naturally isomorphic to $C_{x_0}(X)$ by means of the mapping that sends the equivalence class $[f] \in \widehat{C}(X)$ to the function $f - f(x_0) \in C_{x_0}(X)$. Composing with this isomorphism, we obtain the continuous injection

$$\iota_{x_0} : X \to C_{x_0}(X)$$

given by

$$\iota_{x_0}(z)(\cdot) = d(\cdot, z) - d(x_0, z).$$

It is then clear that the horofunction extension $\overline{X}^h$ of X can be canonically identified with the closure of $\iota_{x_0}(X)$ in $C_{x_0}(X)$, which in particular does not depend on the chosen base point x_0. Note that, for each $z \in X$, the function

$$x \mapsto \iota_{x_0}(z)(x) = d(x,z) - d(x_0,z)$$

is 1-Lipschitz and satisfies that

$$-d(x_0,x) \leq \iota_{x_0}(z)(x) \leq d(x,x_0)$$

for every $x \in X$. This yields that the family of functions $\{\iota_{x_0}(z)(\cdot)\}_{z\in X}$ is equicontinuous and pointwise relatively compact in $C(X)$ and consequently, from Arzela-Ascoli theorem, we deduce that the horofunction extension $\overline{X}^h$ of X is a compact space.

The following subsection provides an alternative way to obtain the same conclusion.

1.2. **Construction using 1-Lipschitz functions.** Let (X,d) be a metric space and consider a fixed base point $x_0 \in X$. We follow here the construction of [16]. Denote by $\mathrm{Lip}^1_{x_0}(X)$ the space of all 1-Lipschitz real-valued functions on X vanishing at x_0. Notice that for every $f \in \mathrm{Lip}^1_{x_0}(X)$ we have:

$$-d(x_0,x) \leq f(x) \leq d(x,x_0), \qquad \text{for all } x \in X.$$

Therefore, identifying f by its values $(f(x))_{x\in X}$ we readily obtain:

$$\mathrm{Lip}^1_{x_0}(X) \subset \prod_{x\in X}[-d(x_0,x), d(x,x_0)] \subset \mathbb{R}^X$$

Notice that by Tychonoff theorem the above product is a compact space. Endowing $\mathrm{Lip}^1_{x_0}(X)$ with the pointwise topology inherited from the Cartesian product $\mathbb{R}^X$, we conclude easily that $\mathrm{Lip}^1_{x_0}(X)$ is closed. As a consequence, $\mathrm{Lip}^1_{x_0}(X)$ is in fact a compact subspace of $\mathbb{R}^X$. On the other hand, it is easily seen that the compact-open topology of $\mathrm{Lip}^1_{x_0}(X)$ coincides with its pointwise topology and $\mathrm{Lip}^1_{x_0}(X)$ is closed in $C_{x_0}(X)$.

Now for each $z \in X$, let us denote by brevity

$$h_z(\cdot) := \iota_{x_0}(z)(\cdot) = d(\cdot,z) - d(x_0,z).$$

Then $h_z \in \mathrm{Lip}^1_{x_0}(X)$. It follows easily that the mapping

$$\begin{cases} h : X \to \mathrm{Lip}^1_{x_0}(X) \subset \mathbb{R}^X \\ h(z) := h_z \end{cases} \tag{1.1}$$

is well-defined and is a continuous injection. In this way we have that the horofunction extension $\overline{X}^h$ of X coincides with the pointwise closure of $h(X)$ in $\mathrm{Lip}^1_{x_0}(X)$, and it is therefore a compact set. As observed by the referee, this conclusion can also be obtained by noticing that $\mathrm{Lip}^1_{x_0}(X)$ is the closed unit ball of a dual space (the space of real-valued Lipschitz functions on X vanishing at x_0 equipped with the Lipschitz norm). In bounded sets of this dual space, the weak*-topology coincides with the topology of pointwise convergence.

Proposition 1.2 (Horofunction extension vs dense subsets)**.** *Let (X,d) be a metric space.*

(i). *If Z is a dense subspace of X, then $\overline{Z}^h$ is homeomorphic to $\overline{X}^h$.*
(ii). *If X is separable, then $\overline{X}^h$ is metrizable.*

Proof. (i). Let Z be dense in X and fix $x_0 \in Z$. It is clear that the natural restriction map

$$r : \mathrm{Lip}^1_{x_0}(X) \to \mathrm{Lip}^1_{x_0}(Z)$$

is a homeomorphism when we consider, respectively, the topology of pointwise convergence on X and the topology of pointwise convergence on Z. We consider, as before, the mapping h given by (1.1) and its restriction to Z:

$$h_{|Z} : Z \to \mathrm{Lip}^1_{x_0}(Z) \subset \mathbb{R}^Z.$$

Since Z is dense in X, we have that the closures of $h(Z)$ and $h(X)$ in $\mathrm{Lip}^1_{x_0}(X)$ coincide. On the other hand, since r is a homeomorphism, the closure of $h_{|Z}(Z)$ in $\mathrm{Lip}^1_{x_0}(Z)$ is $r(\overline{h(Z)})$. Thus the mapping r is a homeomorphism between $\overline{X}^h$ and $\overline{Z}^h$.
(ii). If X is separable, choose a countable dense subspace Z. Then we have

$$h : X \to \mathrm{Lip}^1_{x_0}(X) \approx \mathrm{Lip}^1_{x_0}(Z) \subset \mathbb{R}^Z$$

Since Z is countable, the space $\mathbb{R}^Z$ (equipped with the Cartesian topology) is metrizable. The proof is complete. ■

Remark 1.3 (Completion)**.** *It follows from the above proposition that the horofunction extension of any metric space coincides with the horofunction extension of its completion.*

1.3. **Injection versus embedding.** Let (X, d) be a metric space. We already saw that the horofunction extension $\overline{X}^h$ is a compact space and the injection $h : X \to h(X) \subset \overline{X}^h$ is continuous. Nevertheless $\overline{X}^h$ is not, in general, a *compactification* of X: indeed, this would require $h : X \to h(X) \subset \overline{X}^h$ to be a *topological embedding*, that is, h to be a homeomorphism from X to $h(X)$. This is not always the case, even if the space X is *proper* (that is, every closed bounded subset of X is compact). To illustrate this, we give the following example:

Example 1.4. Consider the Banach space $\ell^1(\mathbb{N})$ endowed with its usual norm, given by $\|x\| = \sum_{k=1}^{\infty} |x_k|$, for every sequence $x = (x_k) \in \ell^1(\mathbb{N})$. For $n = 0$, set $z_0 = 0$, and for each $n \geq 1$, set $z_n = ne_n$, where $\{e_n\}$ denotes the unit vector basis of $\ell^1(\mathbb{N})$. Now, for $n \geq 0$, consider the closed segment $S_n := [z_n, z_{n+1}]$, and define the ℓ^1*-ray* by

$$X := \bigcup_{n\geq 0} S_n \tag{1.2}$$

with the metric inherited from the ℓ^1-norm.

Notice that for any $x \in S_n$, we have $x = (1-t)ne_n + t(n+1)e_{n+1}$ for some $0 \leq t \leq 1$, yielding $\|x\| \geq n$. As a consequence, denoting by $\overline{B}_k$ the closed ball in $\ell^1(\mathbb{N})$ centered at 0 with radius $k \in \mathbb{N}$, we deduce that $X \cap \overline{B}_k$ is contained in $S_1 \cup \cdots \cup S_k$, which is

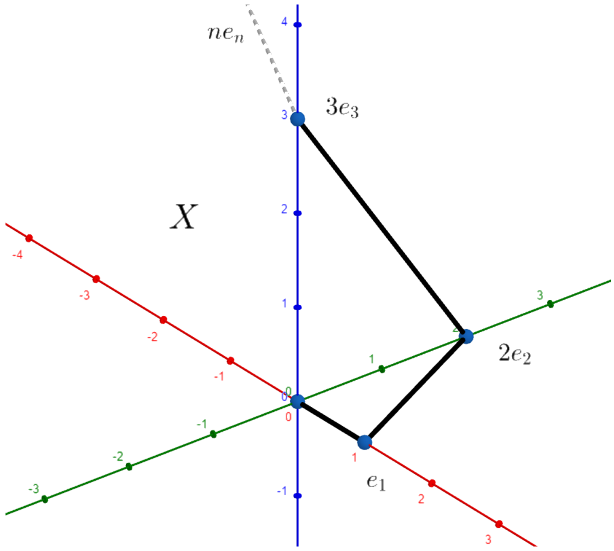

FIGURE 1. Representation of the space (inside ℓ^1).

compact. Therefore X is a proper metric space.

Let us now choose $x_0 = 0$ as a base point and consider the corresponding mapping

$$\begin{cases} h : X \to \mathbb{R}^X \\ z \mapsto h_z(\cdot) := \| \cdot - z\| - \|z\| \end{cases}$$

for every $z \in X$. In order to see that h is not a topological embedding, we consider the sequence $(z_n)_{n\geq 0} \subset X$, where $z_n = ne_n$. It is clear that $(z_n)_{n\geq 0}$ does not converge to $z_0 = 0$ in X. Nevertheless from the following claim we have that the sequence of functions (h_{z_n}) converges pointwise on X to the function $h_0 = \| \cdot \|$.

Claim. Let (a_n) be a sequence in X such that $(\|a_n\|) \to \infty$. Then the sequence (h_{a_n}) converges pointwise on $\ell^1(\mathbb{N})$ to the function $h_0 = \| \cdot \|$.

Proof of the claim. Indeed, let us denote by $c_{00}(\mathbb{N})$ the space of eventually null sequences, that is, $\bar{x} \in c_{00}(\mathbb{N})$ if and only if $\bar{x}$ has finite nonzero terms. Since $c_{00}(\mathbb{N})$ is dense in $\ell^1(\mathbb{N})$, fixing $x \in \ell^1(\mathbb{N})$ and $\varepsilon > 0$, we can find $\bar{x} = (\bar{x}_i) \in c_{00}(\mathbb{N})$ such that $\|x - \bar{x}\| \leq \varepsilon$. Let $k \in \mathbb{N}$ be such that $\bar{x}_i = 0$ for all $i \geq k$ and let $n_0 \in \mathbb{N}$ be such that $a_n \notin S_1 \cup \cdots \cup S_k$, for every $n \geq n_0$ (this is possible since $(a_n) \subset X$ and $\|a_n\| \to \infty$). It follows that for $n \geq n_0$, the sequences a_n and $\bar{x}$ have disjoint supports. Therefore

$$\|\bar{x} - a_n\| = \|\bar{x}\| + \|a_n\|.$$

Then

$$h_{a_n}(x) - h_0(x) = \|x - a_n\| - \|a_n\| - \|x\| \leq \|x - \bar{x}\| + \|\bar{x} - a_n\| - \|a_n\| - \|x\| \leq \varepsilon + \|\bar{x}\| - \|x\| \leq 2\varepsilon$$

and

$$h_{a_n}(x) - h_0(x) = \|x - a_n\| - \|a_n\| - \|x\| \geq \|\bar{x} - a_n\| - \|x - \bar{x}\| - \|a_n\| - \|x\| \geq -2\varepsilon.$$

Let us finally describe the horofunction extension of the metric space X given in (1.2). From Proposition 1.2 we have that $\overline{X}^h$ is metrizable, so every function $f \in \overline{X}^h$ is the pointwise limit of a sequence in $h(X)$. Let (a_n) be a sequence in X such that h_{a_n} converges to f. If (a_n) is bounded, it admits a subsequence (a_{n_j}) convergent to some $a \in X$. Then $(h_{a_{n_j}})$ (and thus (h_{a_n})) converges to h_a, so $f = h_a$. Otherwise, if (a_n) is not bounded, it admits a subsequence (a_{n_j}) such that $(\|a_{n_j}\|)$ converges to ∞, so by the previous claim we have that (h_{a_n}) converges to h_0. This shows that $\overline{X}^h = h(X)$ (as a set) and the horofunction boundary $\overline{X}^h \setminus h(X)$ is empty. Let us finally notice that X is homeomorphic to the ray $[0, +\infty)$ and $\overline{X}^h$ is homeomorphic to the circle $\mathbb{S}^1$.

Remark 1.5. *Using the previous claim and a similar argument, we can see that also for the space $X = \ell^1(\mathbb{N})$, the injection $h : X \to \mathbb{R}^X$ is not a topological embedding. Therefore, the Banach space $\ell^1(\mathbb{N})$ is not Gromov-compactifiable. We shall see in Section 3 that $\ell^1(\mathbb{N})$ represents a prototype of pathology for normed spaces.*

It is well-known that every finite dimensional normed space is Gromov-compactifiable. Indeed, from [11, Lemma 2.2] we have

Proposition 1.6. *Let X be a proper metric space such that every ball is path-connected. Then, $h : X \to \mathbb{R}^X$ is a topological embedding.*

In Section 3, we shall see that the horofunction extension is a compactification for all reflexive Banach spaces (thus for all ℓ_p spaces, with $1 < p < \infty$). For a description of horofunctions of Hilbert space and ℓ_p spaces, we refer to Gutiérrez [17, 16].

We shall now give a topological description of the horofunction extension of the *sphere* of a Hilbert space. Notice that, thanks to the Kadets-Klee property, the topology on the sphere inherited from (the norm-topology of) the Hilbert space coincides with the weak topology.

Example 1.7. *Let $\mathcal{H}$ be a real infinite-dimensional Hilbert space and let us denote by $X = S_{\mathcal{H}}$ its unit sphere. Then the horofunction extension $\overline{X}^h$ is a compactification of $S_{\mathcal{H}}$, homeomorphically equivalent to the closed unit ball $\overline{B}_{\mathcal{H}}$ endowed with the weak topology.*

Proof. Fix a base point $x_0 \in S_{\mathcal{H}}$ and let $z \in \overline{B}_{\mathcal{H}}$. Since the unit sphere $S_{\mathcal{H}}$ is dense in $\overline{B}_{\mathcal{H}}$ for the weak topology, there exists a net $(z_\lambda) \subset S_{\mathcal{H}}$ weakly convergent to z. Then for each $x \in S_{\mathcal{H}}$ the net

$$h_{z_\lambda}(x) := \|x - z_\lambda\| - \|x_0 - z_\lambda\| \underbrace{=}_{\|z_\lambda\|=\|x\|=\|x_0\|=1} \sqrt{2 - 2\langle x, z_\lambda\rangle} - \sqrt{2 - 2\langle x_0, z_\lambda\rangle}$$

converges to the function

$$\psi_z(x) := \sqrt{2 - 2\langle x, z\rangle} - \sqrt{2 - 2\langle x_0, z\rangle},$$

yielding that ψ_z is a horofunction of $S_{\mathcal{H}}$. Therefore the map

$$\begin{cases} \psi : (\overline{B}_{\mathcal{H}}, \text{weak}) \to \overline{X}^h \\ z \mapsto \psi(z)(\cdot) := \psi_z(\cdot) = \sqrt{2-2\langle \cdot, z\rangle} - \sqrt{2-2\langle x_0, z\rangle} \end{cases} \tag{1.3}$$

is well-defined. It is clear that ψ is continuous and ψ_z coincides with h_z whenever $z \in S_{\mathcal{H}}$.

Claim 1: The function ψ is injective.

Proof of Claim 1. Suppose that $\psi(z) = \psi(z')$, where $z, z' \in \overline{B}_{\mathcal{H}}$. Then for each $x \in S_{\mathcal{H}}$,

$$\sqrt{2-2\langle x, z\rangle} - \sqrt{2-2\langle x_0, z\rangle} = \sqrt{2-2\langle x, z'\rangle} - \sqrt{2-2\langle x_0, z'\rangle}$$

and choosing $x \in \{z, z'\}^{\perp}$ we deduce that $\sqrt{2-2\langle x_0, z\rangle} = \sqrt{2-2\langle x_0, z'\rangle}$ and conclude that for every $x \in S_{\mathcal{H}}$,

$$\sqrt{2-2\langle x, z\rangle} = \sqrt{2-2\langle x, z'\rangle}.$$

Therefore

$$\langle x, z\rangle = \langle x, z'\rangle,$$

for every $x \in S_{\mathcal{H}}$. As a consequence, we obtain that $z = z'$.

Claim 2: The function ψ is surjective.

Proof of Claim 2. Consider an element $f \in \overline{X}^h$. Then there is a net $(z_\lambda) \subset S_{\mathcal{H}}$ such that the net (h_{z_λ}) converges to f pointwise on $S_{\mathcal{H}}$. By the weak compactness of the closed ball, there is a subnet (z'_β) weakly convergent to some point $z \in \overline{B}_{\mathcal{H}}$. Then $(h_{z'_\beta})$ converges pointwise on $S_{\mathcal{H}}$ to f and also to ψ_z, so $f = \psi_z$.

Since $(\overline{B}_{\mathcal{H}}, \text{weak})$ is compact, it follows from a standard argument that ψ is a homeomorphism. Since $\psi|_{S_{\mathcal{H}}} = h$, it follows that $S_{\mathcal{H}} = \psi^{-1}(h(S_{\mathcal{H}}))$ and the horofunction extension $\overline{X}^h$ of $S_{\mathcal{H}}$ is a compactification (homeomorphic to $(\overline{B}_{\mathcal{H}}, \text{weak})$). This completes the proof. ■

In the light of the previous examples, a natural question appears: characterize the metric spaces (X, d) for which the horofunction extension $\overline{X}^h$ is a compactification of X, that is, what we have called in Definition 1.1 Gromov-compactifiability.

2. Main results in metric spaces

In this section we establish a characterization of the case when the horofunction extension is a compactification, in the general setting of metric spaces. Some applications are given for specific types of metric spaces. In particular, simplified characterizations are obtained for the cases of locally compact or proper metric spaces, extending previously known results. Finally, we compare the horofunction extension of a locally compact space with the Rieffel *metric compactification* of the space, introduced in [23].

2.1. **A general characterization and first consequences.** We start with a general purely metric characterization of metric spaces X which are Gromov-compactifiable.

Theorem 2.1 (Characterization of Gromov-compactifiability in metric spaces)**.** *Let (X,d) be a metric space. The following conditions are equivalent:*

- (a) *The horofunction extension $\overline{X}^h$ is a compactification of X.*
- (b) *For every point $x \in X$ and every $r > 0$, there exist some $\eta_r > 0$ and a compact set $K_r \subset X$ such that, for each $z \in X \setminus \overline{B}(x,r)$ there exists $w \in K_r$ with*

$$d(w,z) \leq d(w,x) + d(x,z) - \eta_r.$$

Proof. Let us fix $x_0 \in X$ to be a base point for X. Therefore, for every $z \in X$ we have

$$h_z(\cdot) := d(\cdot, z) - d(x_0, z).$$

$(\mathbf{b}) \Rightarrow (\mathbf{a})$. We proceed towards a contradiction, that is, we assume that (b) holds true but the (continuous injective) function $h : X \to h(X) \subset \overline{X}^h$ is not bicontinuous, that is, h^{-1} is not continuous. Therefore, there exist a net $(z_\lambda)_{\lambda\in\Lambda} \subset X$ and $x \in X$ such that

$$(h_{z_\lambda}) \to h_x \text{ uniformly in compact sets, but } (z_\lambda) \not\to x.$$

Therefore, there is $r > 0$ such that the set

$$\Lambda_0 := \{\lambda \in \Lambda : \ d(z_\lambda, x) \geq r\}$$

is a cofinal of Λ. Thus, $(z_\lambda)_{\lambda\in\Lambda_0}$ is a subnet of $(z_\lambda)_{\lambda\in\Lambda}$. Fix $\eta_r > 0$ and the compact set K_r given by statement (b). Consider now the set

$$\Lambda_1 := \{\lambda \in \Lambda_0 : \ z_\lambda \notin K_r\}$$

Claim: Λ_1 is a cofinal of Λ_0 and therefore, $(z_\lambda)_{\lambda\in\Lambda_1}$ is a subnet of $(z_\lambda)_{\lambda\in\Lambda}$.

Proof of the claim: Indeed, otherwise the set $\Gamma_1 := \Lambda_0 \setminus \Lambda_1$ is a cofinal of Λ_0. Since $(z_\lambda)_{\lambda\in\Gamma_1} \subset K_r$, by compactness there is a subnet $(z_\beta)_{\beta\in\Gamma_2}$ convergent to some point $z \in K_r$. Note that $z \neq x$. Since the mapping $h : X \to \overline{X}^h$ is continuous, we have that

$$h_x = \lim_{\lambda\in\Lambda} h_{z_\lambda} = \lim_{\beta\in\Gamma_2} h_{z_\beta} = h_z$$

Since h is injective, we get a contradiction. This completes the proof of the claim.

For any $\lambda \in \Lambda_1$, let $w_\lambda \in K_r$ be the point given by statement (b) associated to z_λ, i.e.

$$d(w_\lambda, z_\lambda) \leq d(w_\lambda, x) + d(x, z_\lambda) - \eta_r, \text{ for all } \lambda \in \Lambda_1.$$

Since $(h_{z_\lambda})_{\lambda\in\Lambda_1}$ converges to h_x uniformly on compact sets, it converges uniformly on $K_r \cup \{x\}$. So, we have that

$$h_x(x) - h_{z_\lambda}(x) = -d(x_0, x) - d(x, z_\lambda) + d(x_0, z_\lambda) =: \alpha_\lambda \to 0.$$

However,

$$\begin{aligned} h_x(w_\lambda) - h_{z_\lambda}(w_\lambda) &= d(w_\lambda, x) - d(x_0, x) - d(w_\lambda, z_\lambda) + d(x_0, z_\lambda) \\ &= \alpha_\lambda + d(x, z_\lambda) + d(w_\lambda, x) - d(w_\lambda, z_\lambda) \\ &\geq \alpha_\lambda + \eta_r \to \eta_r > 0. \end{aligned}$$

This contradicts the fact that $(h_{z_\lambda})_{\lambda \in \Lambda_1}$ converges to h_x uniformly on $K_r \cup \{x\}$.

(**a**) $\Rightarrow$ (**b**). If X is compact, the result follows trivially by choosing, given $r > 0$, $\eta_r = r$, $K_r = X$ and $w = z$. If X is not compact, we proceed by a contrapositive argument, that is, we assume that (b) does not hold and we prove that h^{-1} is not continuous. Since X is not compact, then it is not pseudocompact and there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) > 0$ for all $x \in X$ and $\inf_X f = 0$ (see e.g. [8]). Let us define

$$\mathcal{K} := \{K \subset X : \ K \text{ nonempty compact}\}$$

and the partial order $\leq$ on $\mathcal{K}$ given by the set inclusion:

$$\text{for all } K_1,\ K_2 \in K, \qquad K_1 \leq K_2 \Leftrightarrow K_1 \subset K_2.$$

Consider now the net $(\eta_K)_{K \in \mathcal{K}} \subset \mathbb{R}$ defined by

$$\eta_K := \min\{f(x) : \ x \in K\} > 0, \text{ for all } K \in \mathcal{K}.$$

Since $\inf_X f = 0$, it follows that $(\eta_K)_{K \in \mathcal{K}}$ converges to 0. Choose $x \in X$ and $r > 0$ for which the statement (b) does not hold. Then for each compact set $K \in \mathcal{K}$ there is some $z_K \in X \setminus \overline{B}(x, r)$ satisfying

$$d(w, z_K) > d(w, x) + d(x, z_K) - \eta_K, \text{ for all } w \in K.$$

We show that $(h_{z_K})_{K \in \mathcal{K}}$ converges to h_x uniformly on compact sets, but $(z_K)_{K \in \mathcal{K}}$ does not converge to x. The second part is clear from the fact that $d(x, z_K) \geq r > 0$ for all $K \in \mathcal{K}$. Now fix $L_0 \in \mathcal{K}$. Then, for any $L \in \mathcal{K}$ such that $L \supset L_0 \cup \{x_0\}$ we have that

$$\begin{aligned} |h_{z_L}(x) - h_x(x)| &= |d(x, z_L) - d(x_0, z_L) - d(x, x) + d(x_0, x)| \\ &= d(x_0, x) + d(x, z_L) - d(x_0, z_L) =: \alpha_L < \eta_L. \end{aligned}$$

Observe that the above inequality follows from the fact that $x_0 \in L$. Now, for any $w \in L_0$, we have that

$$\begin{aligned} |h_{z_L}(w) - h_x(w)| &= |d(w, z_L) - d(x_0, z_L) - d(w, x) + d(x_0, x)| \\ &= |d(w, z_L) - d(w, x) + \alpha_L - d(x, z_L)| \\ &\leq d(w, x) + d(x, z_L) - d(w, z_L) + \alpha_L \leq 2\eta_L. \end{aligned}$$

Therefore, we have shown that for any $L \geq L_0 \cup \{x_0\}$,

$$\sup\{|h_x(w) - h_{z_L}(w)| : \ w \in L_0\} \leq 2\eta_L \to 0.$$

Since L_0 is an arbitrary compact subset of X, we have that $(h_{z_K})_{K \in \mathcal{K}}$ converges to h_x uniformly on compact sets. Therefore, $h^{-1} : h(X) \to X$ is not continuous. ■

As usual, the Lipschitz property allows us to replace compact sets by finite sets.

Remark 2.2. Statements (a) and (b) of Theorem 2.1 are also equivalent to the following:

(c) For every $x \in X$ and $r > 0$, there exist $\eta_r > 0$ and a finite set $K_r \subset X$ such that for every $z \in X \setminus \overline{B}(x,r)$ there exists $w \in K_r$ satisfying

$$d(w,z) \le d(w,x) + d(x,z) - \eta_r.$$

Indeed, $(c) \Rightarrow (b)$ follows readily (since every finite set is compact). Assume now that (b) holds, fix $x \in X$, $r > 0$ and let $\eta > 0$ and $K \subset X$ given by statement (b). Since K is compact, there exists a finite set $A \subset K$ which is an $\eta/3$-net of K. For any $w \in K$, take $a_w \in A$ such that $d(w, a_w) \le \eta/3$. Let $z \in X \setminus \overline{B}(x,r)$. Then there is $w \in K$ such that $d(w,z) \le d(w,x) + d(x,z) - \eta$. Therefore $d(a_w,z) \le d(a_w,x) + d(x,z) - \dfrac{\eta}{3}$. Setting $\eta_r := \frac{\eta}{3}$ and $K_r = A$, we see that (c) holds true.

A very interesting consequence of the above characterization is the following result, which provides a completely new insight to the situation observed in Example 1.7.

Corollary 2.3 (Gromov-compactifiability of the sphere of any normed space)**.** *Let (M,d) be a bounded metric space such that for every $x \in M$ we have:*

$$\sup_{y \in M} d(y,x) = \operatorname{diam}(M) := \sup_{y,z \in M} d(y,z). \tag{2.1}$$

Then M is Gromov-compactifiable.

In particular, the unit sphere S_X of any normed space X (equipped with the distance inherited by the norm) is always Gromov-compactifiable.

Proof. Assume that the metric space (M,d) is bounded and satisfies (2.1). We shall show that condition (b) of Theorem 2.1 is fulfilled.

To this end, let $x \in M$ and $r > 0$. If $r \ge \operatorname{diam}(M)$, the conclusion of (b) is vacuously satisfied. Therefore, we may assume that $r < \operatorname{diam}(M)$. Then we fix $\eta = r/2$ and choose $y \in M$ such that

$$d(y,x) \ge \operatorname{diam}(M) - \frac{r}{2}.$$

We set $K = \{y\}$ and observe that for any $z \in M \setminus \overline{B}(x,r)$ we have

$$d(y,x) + d(x,z) \ge \Big(\operatorname{diam}(M) - \frac{r}{2}\Big) + r \ge d(y,z) + \eta.$$

The second part of the statement is straightforward, since for the metric space $M = S_X$ (unit sphere of a normed space X) and for any $x \in S_X$, we can take $y := -x \in S_X$ and observe that

$$d(x,-x) = 2 = \operatorname{diam}(S_X).$$

The proof is complete. ∎

Let us notice that there are simple examples of *bounded* metric spaces which are not Gromov-compactifiable. For example, consider $M = \{e_n : n \in \mathbb{N}\} \cup \{0\}$ as a subspace of $\ell^1(\mathbb{N})$. Here, it is easily seen that the sequence (h_{e_n}) converges to h_0 pointwise on M.

In what follows, we study special classes of metric spaces that allow to simplify the statement of Theorem 2.1 (necessary and sufficient condition for Gromov-compactifiability). Let us start with the following consequence for proper metric spaces, which improves the sufficient condition given in [11, Lemma 2.2].

Corollary 2.4 (Simplified characterization for proper spaces)**.** *Let (X, d) be a proper metric space. The following are equivalent:*

- (a) *The horofunction extension $\overline{X}^h$ is a compactification of X.*
- (b') *For every point $x \in X$, there exist constants $\eta > 0$ and $R > 0$ such that, for each $z \in X \setminus \overline{B}(x, R)$, there exists some $w \in \overline{B}(x, R)$ such that*

$$d(w, z) \leq d(w, x) + d(x, z) - \eta.$$

Proof. First note that condition (b') above implies condition (b) of Theorem 2.1. Indeed, if for every $x \in X$ we have constants $\eta > 0$ and $R > 0$ satisfying (b') it is clear that condition (b) is fulfilled if for each $r > 0$ we choose $\eta_r := \eta$ and $K_r := \overline{B}(x, R)$.

Conversely, choose any $r > 0$, e.g. $r = 1$, then get η_r and K_r from condition (b) of Theorem 2.1. Now take $R > r$ so that $B(x, R)$ contains K_r. Then (b') follows at once.

■

We now obtain some sufficient conditions in the setting of locally compact metric spaces. As in the previous case, the first one is a direct consequence of Theorem 2.1.

Corollary 2.5 (Locally compact spaces)**.** *Let (X, d) be a locally compact metric space. Suppose that for every point $x \in X$, there exist constants $\eta > 0$ and $R > 0$ such that the ball $\overline{B}(x, R)$ is compact, and for each $z \in X \setminus \overline{B}(x, R)$, there exists some $w \in \overline{B}(x, R)$ such that*

$$d(w, z) \leq d(w, x) + d(x, z) - \eta.$$

Then the horofunction extension $\overline{X}^h$ is a compactification of X.

The following corollary is a generalization of Proposition 1.6.

Corollary 2.6. *Let (X, d) be a locally compact metric space such that every ball in X is connected. Then the horofunction extension $\overline{X}^h$ is a compactification of X.*

Proof. It suffices to show that the condition of Corollary 2.5 is fulfilled. Given $x \in X$, we can choose any $R > 0$ such that $\overline{B}(x, R)$ is compact, and $\eta = \frac{R}{2}$. Indeed, for each

$z \in X \setminus \overline{B}(x,R)$, set $R' = d(x,z) > R$ and consider the closed ball $\overline{B}(z,R')$. Since this ball is connected, the set

$$F := \left\{ y \in \overline{B}(z,R') \,:\, d(y,x) = \frac{R}{2} \right\}$$

is nonempty. Choosing $w \in F$ we obtain that $d(w,z) \leq R'$ and

$$d(x,z) + d(w,x) - d(w,z) \geq R' + \frac{R}{2} - R' = \frac{R}{2}.$$

The proof is complete. ■

In the previous result, local compactness is an important assumption. Indeed, in Remark 1.5 we saw that for the space $X = \ell^1(\mathbb{N})$, the horofunction extension $\overline{X}^h$ is not a compactification of X, although $\ell^1(\mathbb{N})$ is a geodesic space.

In fact, without local compactness, we cannot ensure a positive result even for metric trees. Recall that a metric space (X,d) is said to be a *metric tree* or $\mathbb{R}$-*tree* if it satisfies the following two conditions:

(i). for every $x, y \in X$, there exists a unique geodesic segment $[x,y]$ joining them, and

(ii). If $[y,x] \cap [x,z] = \{x\}$ then $[y,x] \cup [x,z] = [y,z]$.

Example 2.7 (Non-locally compact metric tree)**.** In the Banach space $\ell^1(\mathbb{N})$ consider the union of segments

$$X := \bigcup_{n=1}^{\infty} [0, ne_n].$$

The space X, endowed with the metric inherited from $\ell^1(\mathbb{N})$, is a metric tree. Evoking again the claim of Example 1.4 we deduce that $\overline{X}^h$ is not a compactification of X.

Let us finish this subsection with the following application to ultrametric spaces. Recall that a metric space (X,d) is called *ultrametric* if, for every $x, y, z \in X$,

$$d(x,z) \leq \max\{d(x,y), d(y,z)\}.$$

Corollary 2.8 (Ultrametric spaces are Gromov-compactifiable)**.** *Let (X,d) be an ultrametric space. Then the horofunction extension $\overline{X}^h$ is a compactification of X.*

Proof. Let us check that Theorem 2.1 (b) holds true. Let $x \in X$ and $r > 0$. Assume that $X \setminus \overline{B}(x,r) \neq \emptyset$ and set $K = \{w\}$, where $d(x,w) > r$. We show that the choice $\eta_r = r$ satisfies (b) of Theorem 2.1. Indeed, for any $z \in X \setminus \overline{B}(x,r)$, we have that

$$d(z,w) \leq \max\{d(z,x), d(x,w)\} \leq d(z,x) + d(x,w) - r.$$

The proof is complete. ■

2.2. **Alternative constructions.** We recall the classical construction of extensions of a metric (or topological) space X by using a family of continuous *bounded* functions on X (see, e.g. Chandler [4]). In our case, let (X,d) be a metric space, and let $\mathcal{L}$ be a family of continuous bounded real functions on X, which separates the points of X. Consider the injection

$$e_{\mathcal{L}} : X \to \mathbb{R}^{\mathcal{L}}$$

defined by

$$e_{\mathcal{L}}(z) := (f(z))_{f\in\mathcal{L}}\,.$$

The the *associated extension* $H_{\mathcal{L}}(X)$ of X is defined as the closure of $e_{\mathcal{L}}(X)$ in $\mathbb{R}^{\mathcal{L}}$, when this space is endowed with the product topology. It is easily seen that $e_{\mathcal{L}}$ is a continuous injection and $H_{\mathcal{L}}(X)$ is compact. Note that

$$e_{\mathcal{L}}(X) \subset \prod_{f\in\mathcal{L}}[\inf_{z\in X} f(z), \sup_{z\in X} f(z)]$$

Furthermore, it is well-known that $e_{\mathcal{L}}$ is a topological embedding if, and only if, the family $\mathcal{L}$ *weakly separates points and closed sets* of X. This means (see, e.g. [5]) that for every $z_0 \in X$ and every closed set F in X with $z_0 \notin F$, there exist $f_1,\ldots,f_m \in \mathcal{L}$ such that $0 \notin \overline{g(F)}$, where $g : X \to \mathbb{R}$ is defined by

$$g(z) := \max_{1\le k\le m} |f_k(z) - f_k(z_0)|$$

In this case, $H_{\mathcal{L}}(X)$ is a compactification of X and, for each $f \in \mathcal{L}$, the natural projection π_f provides a continuous extension of f to $H_{\mathcal{L}}(X)$. In fact, $H_{\mathcal{L}}(X)$ can be characterized as the smallest compactification of X where every function in $\mathcal{L}$ can be continuously extended (see [4]). Here, we consider the *usual ordering* in the family of compactifications of X. That is, for two compactifications $\alpha_1 X$ and $\alpha_2 X$ of X, we say that $\alpha_1 X \le \alpha_2 X$ whenever there exists a continuous map $\varphi : \alpha_2 X \to \alpha_1 X$ leaving X pointwise fixed. We also say that $\alpha_1 X$ and $\alpha_2 X$ are *equivalent* if $\alpha_1 X \le \alpha_2 X$ and $\alpha_2 X \le \alpha_1 X$. This implies the existence of a homeomorphism $\varphi : \alpha_1 X \to \alpha_2 X$ leaving X pointwise fixed.

Now fix a base point $x_0 \in X$ and, for each $x \in X$, consider the function $\theta_x : X \to \mathbb{R}$ defined as

$$\theta_x(z) := d(x,z) - d(x_0,z).$$

Note that θ_x is a bounded 2-Lipschitz function on X and, for every $x,z \in X$:

$$\theta_x(z) = h_z(x).$$

Further, consider the family

$$\mathcal{L}_\theta := \{\theta_x \,:\, x \in X\}. \tag{2.2}$$

Then we have the following.

Proposition 2.9. *Let (X,d) be a metric space. Then* (i). *The horofunction extension* $\overline{X}^h$ *coincides with the extension* $H_{\mathcal{L}_\theta}(X)$.

(ii). *The horofunction extension* $\overline{X}^h$ *is a compactification of* X *if, and only if, the family* $\mathcal{L}_\theta$ *weakly separates points and closed sets of* X.

Proof. Note that, for every $z \in X$ we can identify:

$$e_{\mathcal{L}_\theta}(z) = (\theta_x(z))_{x\in X} = (h_z(x))_{x\in X} = h_z.$$

From this, part (i) follows at once. On the other hand, as we have mentioned, $e_{\mathcal{L}}$ is a topological embedding if, and only if, the family $\mathcal{L}$ weakly separates points and closed sets of X, so (ii) follows. ■

The Rieffel construction. In the case that (X, d) is a locally compact metric space, Rieffel defines the *metric compactification* $\overline{X}^d$ of X (see Definition 4.1. in [23]) as the maximal ideal space of the uniformly closed algebra of (bounded) functions on X generated by the union of the family $\mathcal{L}_\theta$ (given in (2.2)), the family of constant functions, and the family $C_\infty(X)$ of all continuous functions on X vanishing at infinity. Recall that $f \in C_\infty(X)$ if, and only if, for every $\varepsilon > 0$ there is a compact set K such that $|f(x)| < \varepsilon$ whenever $x \notin K$. It is clear that this metric compactification can also be obtained following our previous scheme, and in fact $\overline{X}^d = H_{\mathcal{L}_d}(X)$, where

$$\mathcal{L}_d := \mathcal{L}_\theta \cup C_\infty(X).$$

Note that $\overline{X}^d$ is always a compactification of X, so in general it can be different from $\overline{X}^h$ (see Example 1.4). Nevertheless, we always have the natural projection map

$$\pi : \overline{X}^d = H_{\mathcal{L}_d}(X) \to \overline{X}^h \subset H_{\mathcal{L}_\theta}(X)$$

which is continuous, closed and surjective, and satisfies $\pi(z) = h_z$ for every $z \in X$. If $\overline{X}^h$ is a compactification of X, the above map π gives that $\overline{X}^h \leq \overline{X}^d$ with the usual ordering. Furthermore, in this case each function in $\mathcal{L}_d$ extends continuously to $\overline{X}^h$, since every function in $C_\infty(X)$ can be continuously extended to every compactification of X, by assigning the value 0 outside X. Therefore, from the minimality of $H_{\mathcal{L}_d}(X)$ with respect to this property, we obtain that $\overline{X}^d \leq \overline{X}^h$.

Summarizing, we obtain:

Proposition 2.10. *Let* (X, d) *be a locally compact metric space.*

(i). *The horofunction extension* $\overline{X}^h$ *is a quotient of the metric compactification* $\overline{X}^d$.

(ii). *If the horofunction extension* $\overline{X}^h$ *is a compactification of* X, *then* $\overline{X}^d = \overline{X}^h$.

As an illustrative example, consider X to be the ℓ^1-ray defined in Example 1.4. Here, X is a locally compact metric space homeomorphic to $[0, +\infty)$ whose one-point compactification is $[0, +\infty]$. In addition, if we fix $x_0 = 0$, we obtain from the Claim in Example 1.4

that, for every $x \in X$:

$$\lim_{\|z\|\to\infty} \theta_x(z) = \lim_{\|z\|\to\infty} (\|x-z\| - \|z\|) = \|x\|.$$

This implies that θ_x extends continuously to the one-point compactification of X. Therefore $\overline{X}^d$ coincides with this compactification, that is, $\overline{X}^d = [0,+\infty]$. On the other hand, as we have seen in Example 1.4, we have that $\overline{X}^h \setminus X$ is empty and $\overline{X}^h = \mathbb{S}^1$, where the natural quotient map

$$\pi : \overline{X}^d = [0,+\infty] \to \overline{X}^h = \mathbb{S}^1$$

identifies 0 with the point at infinity.

Open question. It would be interesting to know if every non-Gromov-compactifiable metric space (X,d) admits a *minimal* compactification with the property that every function in the family $\mathcal{L}_\theta$ can be continuously extended there. If (X,d) is locally compact, the answer is positive, since the metric compactification $\overline{X}^d$ considered above has this property.

3. Main results in normed spaces

In this section we establish a characterization of Gromov-compactifiability in the setting of normed spaces. Here, the richer structure of the space allows for a more geometric characterization, in terms of the Hausdorff distance between spheres. Furthermore, we obtain a connection with the so-called *octahedrality* of the norm, a property introduced in [13] and well studied in the realm of Banach space geometry. This allows us to characterize the stability of Gromov-compactifiability under renormings, in terms of non-containment of an isomorphic copy of ℓ^1. We finish with some applications to Lipschitz-free spaces and 1-Wasserstein spaces.

3.1. **A simplified characterization for normed spaces.** We now provide several applications of Theorem 2.1 for (infinite dimensional) normed spaces. We start with the a general characterization. In what follows, $d_H(A,B)$ stands for the Hausdorff-Pompeiu distance between two subsets A and B of a metric space (X,d), that is,

$$d_H(A,B) := \max\left\{\sup_{x\in A} d(x,B),\ \sup_{x\in B} d(x,A)\right\},$$

where $d(x,C) = \inf_{y\in C} d(x,y)$, for any $x \in X$ and $C \subset X$.
Notice that in the special case that $A \subset B$ we have:

$$d_H(A,B) = \sup_{x\in B}\, d(x,A).$$

In what follows, we shall need the following lemma. The proof follows easily from the triangle inequality.

Lemma 3.1. *Let X be a normed space. For any two vectors $u, v \in X$ and $t \geq 1$ it holds:*

$$\|u\| - \|v-u\| \;\leq\; t\|u\| - \|v-tu\|.$$

The main result of this subsection reads as follows.

Theorem 3.2 (Characterization of Gromov-compactifiability in normed spaces). *For a normed space $(X, \|\cdot\|)$ the following statements are equivalent:*

- (a) *The horofunction extension $\overline{X}^h$ is a compactification of X.*
- (b) *There exist $\eta > 0$, $M > 0$ and a finite dimensional subspace $F \subset X$ such that for every $z \in X \setminus \overline{B}(0,1)$, there is $w \in M\overline{B}_F := \overline{B}(0,M) \cap F$ with*

$$\|z - w\| \leq \|z\| + \|w\| - \eta.$$

- (c) *For some finite dimensional subspace $F \subset X$ we have:*

$$d_H(S_F, S_X) < 2$$

where S_F and S_X denote the unit spheres of F and X respectively.

Proof. Let us first notice that if X is finite dimensional, then all assertions are true. Indeed, Proposition 1.6 yields that $\overline{X}^h$ is a compactification of X, (b) follows easily by taking $F = X$, $r = \eta = 1$ and $w = z/\|z\|$ and (c) follows trivially by choosing $F = X$.

Let us now assume that X is infinite dimensional. We prove the following chain of implications: $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$.

(**a**) $\Rightarrow$ (**b**): It follows directly from Theorem 2.1 and Remark 2.2 (c). Indeed, for $0 \in X$ and $r = 1$, let $K \subset X$ be finite and $\eta > 0$ such that for any $z \in X \setminus \overline{B}(0,1)$, there is $w \in K$ such that $\|w - z\| \leq \|w\| + \|z\| - \eta$. Set $F = \operatorname{span}(K)$ and $M := \max\{\|w\| : \ w \in K\}$.

(**b**) $\Rightarrow$ (**c**): Let $\eta > 0$, $M > 0$ and F finite dimensional subspace of X be given by statement (b). Without loss of generality, we may assume that $M > 1$. Fix $\bar{z} \in S_X$. Since $\|\bar{z}\| = 1$ we have $z := M\bar{z} \in X \setminus \overline{B}(0,1)$ and there exists $w \in M\overline{B}_F$ such that

$$\eta \leq \|z\| + \|w\| - \|z - w\|.$$

Note that $w \neq 0$. Applying Lemma 3.1 for $t = M/\|w\|$, we obtain that

$$\|w\| - \|z - w\| \leq \|tw\| - \|z - tw\| = M - \big\|z - M(w/\|w\|)\big\|.$$

Summing up both last inequalities, we have:

$$\eta \leq 2M - \big\|z - M(w/\|w\|)\big\| \quad \text{and consequently} \quad \frac{\eta}{M} \leq 2 - \big\|\bar{z} - (w/\|w\|)\big\|.$$

Since $w/\|w\| \in S_F$ and above holds true for every $\bar{z} \in S_X$, we deduce that

$$d_H(S_F, S_X) \ \leq \ 2 - \frac{\eta}{M} \ < \ 2\,.$$

(**c**) $\Rightarrow$ (**a**): Since the distance $(d(x_1, x_2) = \|x_1 - x_2\|)$ in a normed space is invariant under translations, we only need to check the statement (b) of Theorem 2.1 for $x = 0$. Let $F \subset X$ be given by statement (c), that is, $\eta < 2 - d_H(S_F, S_X)$ for some $\eta > 0$ and let us denote by K the closed unit ball $\overline{B}_F$ of F, which is a compact set. The implication readily follows from the next claim.

Claim: For any $r > 0$, the statement (b) of Theorem 2.1 is satisfied for $x = 0$ by taking $\eta_r = r\eta$ and $K_r = rK$.

Proof of the claim. Fix $r > 0$. Observe that, for every $z \in X \setminus \overline{B}(0, r)$, we have $r^{-1}z \in X \setminus \overline{B}(0, 1)$. Since $t := (1/r)\|z\| \geq 1$, applying Lemma 3.1 for $u = z/\|z\|$ and $v = w \in S_F$ we obtain:

$$\big\|(z/\|z\|)\big\| - \big\|(z/\|z\|) - w\big\| \leq \big\|(1/r)z\big\| - \big\|(1/r)z - w\big\|.$$

By hypothesis (c), for some $w \in S_F$ we have:

$$\big\|(z/\|z\|) - w\big\| \;\leq\; 2 - \eta \;=\; \big\|(z/\|z\|)\big\| + \|w\| - \eta.$$

Summing up both last inequalities and then multiplying both sides of the resulting inequality by r, we obtain

$$\|z - rw\| \leq \|z\| + \|rw\| - r\eta.$$

Since $rw \in rK = K_r$ and $r\eta = \eta_r > 0$, the result follows. ■

3.2. **Gromov-compactifiability under renormings.** In this subsection we obtain concrete applications of Theorem 3.2 in connection with the geometry and structure of Banach spaces. Let us recall that a Banach space $(X, \|\cdot\|)$ is said to be *octahedral* (see [6, 13] *e.g.*) if, for every $\eta > 0$ and every finite-dimensional subspace F of X, there exists a point $z \in S_X$ such that

$$\|z - w\| \geq (1 - \eta)(1 + \|w\|), \text{ for all } w \in F.$$

In order to connect this property with the conditions of Theorem 3.2, the following characterization of octahedrality, given in [18, Proposition 2.2], will be useful.

Proposition 3.3 (Characterization of octahedrality). *The following assertions are equivalent for a Banach space $(X, \|\cdot\|)$:*

(i). *$(X, \|\cdot\|)$ is octahedral.*

(ii). *For every $\eta > 0$ and every finite set of points $w_1, \ldots, w_n \in S_X$, there exists $z \in S_X$ such that*

$$\|z - w_i\| \geq 2 - \eta, \text{ for all } i \in \{1, \ldots, n\}.$$

Using the above result, we can obtain a further geometric characterization of Gromov-compactifiablity for Banach spaces.

Theorem 3.4 (Characterization by non-octahedrality). *The following assertions are equivalent for a Banach space $(X, \|\cdot\|)$:*

(i). *The horofunction extension $\overline{X}^h$ is a compactification of X.*

(ii). *$(X, \|\cdot\|)$ is not octahedral.*

Proof. Let us assume that (ii) fails, that is, the space $(X, \|\cdot\|)$ is octahedral and let us consider a finite-dimensional subspace F of X. Then for every $\eta > 0$, there exists a point $z = z_\eta \in S_X$ such that

$$\|z_\eta - w\| \geq (1-\eta)(1 + \|w\|), \text{ for every } w \in F.$$

Taking $w \in S_F$ we deduce that $\|z_\eta - w\| \geq 2(1-\eta) = 2 - 2\eta$ and consequently $\operatorname{dist}(z_\eta, S_F) \geq 2 - 2\eta$, yielding $d_H(S_F, S_X) \geq 2$. Therefore, condition (c) of Theorem 3.2 fails, so the horofunction extension $\overline{X}^h$ is not a compactification of X.

Conversely, assume that condition (c) of Theorem 3.2 fails. Then for each finite-dimensional subspace F of X we have that $d_H(S_F, S_X) \geq 2$. Let further $\eta > 0$ and a finite set of points $w_1, \ldots, w_n \in S_X$. Setting $F := \operatorname{span}\{w_1, \ldots, w_n\}$ we deduce that there exists some $z \in S_X$ such that $\operatorname{dist}(z, F) \geq 2 - \eta$. This yields that $\|z - w_i\| \geq 2 - \eta$ for all $i \in \{1, \ldots, n\}$, so from Proposition 3.3 we conclude that $(X, \|\cdot\|)$ is octahedral. ■

In what follows, we are interested in the behavior of the horofunction extension of a normed space under renormings. To this end, let us introduce the following definition.

Definition 3.5 (Stable Gromov-compactification)**.** *A Banach space $(X, \|\cdot\|)$ is said to be stably Gromov-compactifiable if for every equivalent norm $|||\cdot|||$ of $\|\cdot\|$, the horofunction extension $\overline{(X, |||\cdot|||)}^h$ is a compactification of $(X, |||\cdot|||)$.*

We shall also need the following result of Godefroy [13] (see also [6, Theorem III.2.5]) regarding the space $\ell^1 := \ell^1(\mathbb{N})$. We mention for completeness that this result was generalized in [2] for the spaces $\ell^1(\kappa)$.

Theorem 3.6 (Godefroy's characterization of spaces containing ℓ^1)**.** *The following assertions are equivalent for a Banach space $(X, \|\cdot\|)$:*

(i). *X contains an isomorphic copy of ℓ^1.*

(ii). *X admits an equivalent octahedral norm.*

Combining Theorem 3.4 with Theorem 3.6 we obtain readily the following characterization of Gromov-compactifiability under any renorming.

Theorem 3.7 (Gromov-compactifiability under renorming)**.** *Let $(X, \|\cdot\|)$ be a Banach space. The following are equivalent:*

(i). *X does not contain an isomorphic copy of ℓ^1.*

(ii). *X is stably Gromov-compactifiable.*

3.3. **Further applications and an ℓ^1-criterium.** In this section we illustrate our previous results in normed spaces. Theorem 3.7 recovers (and improves) previous results on Gromov-compactifiability for finite normed spaces and for Hilbert spaces mentioned in the introduction. These results are now reinforced, since they hold for any renorming. The same conclusion also applies for the classical ℓ^p-spaces (or more generally $L^p(\Omega,\mu)$), for all $p \in (1,+\infty)$. This is a consequence of the following result.

Corollary 3.8 (Asplund spaces are Gromov-compactifiable). *Asplund spaces (therefore, in particular, reflexive Banach spaces) are stably Gromov-compactifiable.*

Proof. Recall that every reflexive Banach space is Asplund ([9, Corollary 11.10] e.g.). Moreover, if a Banach space X contains an isomorphic copy of ℓ^1, then it contains in particular a separable subspace with a non-separable dual and consequently, X cannot be Asplund ([21, Chapter 5] *e.g.*). We deduce from Theorem 3.7 that every Asplund space is stably Gromov-compactifiable. ■

In the introduction we have seen that the horofunction extension of ℓ^1 is not a topological compactification. In the following result we show that this property also holds for all infinite dimensional L^1-spaces. All these spaces are non-Gromov-compactifiable.

Proposition 3.9 (Examples of non-Gromov-compactifiable spaces). *Let $\{(X_\gamma, \|\cdot\|_\gamma)\}_{\gamma\in\Gamma}$ be an infinite family of normed spaces. Denote by $(X,\|\cdot\|)$ the normed space $(\sum_\gamma X_\gamma)_{\ell_1}$, i.e., the ℓ_1-sum of the spaces $(X_\gamma)_\gamma$*

$$X := \left\{ (x_\gamma)_\gamma \in \prod_{\gamma\in\Gamma} X_\gamma : \ \|(x_\gamma)_\gamma\| := \sum_{\gamma\in\Gamma} \|x_\gamma\|_\gamma < \infty \right\}.$$

Then, the horofunction extension $\overline{X}^h$ is not a compactification of X. In particular, any infinite dimensional $L^1(\Omega,\mu)$ space is not Gromov-compactifiable.

Proof. Let us verify that the space X does not verify the statement (c) of Theorem 3.2. For each $\gamma \in \Gamma$, consider $e_\gamma \in X_\gamma$ be a unit vector. Let $F \subset X$ be any finite dimensional subspace. Since $\overline{B}_F$ is compact, it easily follows that there is a sequence $(\sigma_n)_n \subset [0,\infty)$ and a sequence $(\gamma_n)_n \subset \Gamma$ such that $\lim_{n\to\infty} \sigma_n = 0$ and that

$$\overline{B}_F \subset \prod_{n=1}^{\infty} B_{X_{\gamma_n}}(0,\sigma_n) \times \prod_{\gamma\in\Gamma\setminus\{\gamma_n:\ n\in\mathbb{N}\}} \{0\}.$$

For each $n \in \mathbb{N}$, consider $z_n := e_{\gamma_n}$. Notice that for any $n \in \mathbb{N}$ and any $w \in S_F$, we have

$$\begin{aligned} \|z_n - w\| &= \|e_{\gamma_n} - w_{\gamma_n}\| + \sum_{\substack{\gamma\in\Gamma \\ \gamma\neq\gamma_n}}^{\infty} \|w_\gamma\| \\ &\geq 2 - 2\|w_{\gamma_n}\| \geq 2 - 2\sigma_n. \end{aligned}$$

Since above inequality holds true for any $w \in S_F$, we have shown that

$$d_H(S_F, S_X) \geq 2 - 2\sigma_n.$$

Since n can be taken arbitrarily large, the statement of (c) in Theorem 3.2 is not satisfied.

Let further $(\Omega, \mathcal{A}, \mu)$ be a measure space such that $L^1(\Omega, \mu)$ is infinite dimensional. Then, there is an infinite countable partition of Ω, $\{\Omega_i\}_i \subset \mathcal{A}$, such that $\mu(\Omega_i) > 0$ for all $i \in \mathbb{N}$. The conclusion follows from the following fact:

$$L^1(\Omega, \mu) \text{ is isometrically isomorphic to } \left(\sum_{i=1}^{\infty} L^1(\Omega_i, \mu_i) \right)_{\ell^1},$$

where $\mu_i := \mu|_{\Omega_i}$. ■

Let us now extract the following criterium from Theorem 3.7.

- (ℓ^1-criterium) If the horofunction extension $\overline{X}^h$ of a Banach space X is not a compactification, then X contains an isomorphic copy of ℓ^1.

We shall now show that the converse of the above criterium does not hold, namely, there are Gromov-compactifiable spaces that contain ℓ^1. Notice that this shows in particular that Gromov-compactifiability is not invariant under renormings.

To this end, let $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be two normed spaces. We denote by $Y \oplus_p Z$ the p-sum of Y and Z, where $p \in [1, +\infty]$. That is, the normed space $X := Y \oplus_p Z$ is the direct sum of Y and Z equipped with the norm

$$\|x\| = \|y + z\| := (\|y\|_Y^p + \|z\|_Z^p)^{\frac{1}{p}}, \text{ for all } x \in X,$$

if $p \in [1, +\infty)$ and $\|x\| := \max \{ \|y\|_Y, \|z\|_Z\}$, if $p = +\infty$.

Proposition 3.10. *Let Y, Z be normed spaces. Then:*

(i). *For every $p \in (1, +\infty)$ the space $X = Y \oplus_p Z$ is Gromov-compactifiable.*

(ii). *The space $X = Y \oplus_1 Z$ is Gromov-compactifiable if and only if both spaces Y and Z are Gromov-compactifiable.*

(iii). *If Y is finite dimensional, then the space $X = Y \oplus_\infty Z$ is Gromov-compactifiable.*

(iv). *For every $\Gamma \neq \emptyset$, the spaces $\ell^\infty(\Gamma)$ and $c_0(\Gamma)$ are Gromov-compactifiable.*

Proof. (i). It follows directly from Theorem 3.4 and [18, Proposition 4.7].

(ii). It follows directly from Theorem 3.4 and [22, Proposition 3.7]

(iii). We now consider the case $p = +\infty$ and Y is finite dimensional. We shall show that the statement (c) of Theorem 3.2 holds for $F = Y$. As before, fix $x \in S_X$ and we write $x = x_Y + x_Z$. Therefore, $\|x\| = \max\{\|x_Y\|, \|x_Z\|\}$. If $x_Y = 0$, set $y \in Y$ as any unit vector. If $x_Y \neq 0$, set $y = x_Y / \|x_Y\|$. Observe that, in any case, we have that

$$\|x - y\| = \max\{\|x_Y - y\|, \|x_Z\|\} = \max\{1 - \|x_Y\|, \|x_Z\|\} \leq 1.$$

Since x is arbitrary, we deduce that $d_H(S_F, S_X) = 1$.

(iv). If Γ is a finite set, then $\ell^\infty(\Gamma)$ is finite dimensional. Then, Proposition 1.6 implies that it is Gromov-compactifiable. On the other hand, if Γ is an infinite set, it follows easily that $\ell^\infty(\Gamma)$ and $c_0(\Gamma)$ are isometrically isomorphic to $\mathbb{R} \oplus_\infty \ell^\infty(\Gamma)$ and $\mathbb{R} \oplus_\infty c_0(\Gamma)$ respectively, and consequently (iii) applies. ■

Notice that the above result is sharp in the following sense: the horofunction extension $\overline{X}^h$ of the space $X := \ell^1 \oplus_\infty \ell^1$ is not a compactification of X. Indeed, we consider the (unbounded) sequence $z_n = (ne_n, ne_n) \in X$ and proceed in a similar way as in Example 1.4 to show that the corresponding sequence $\{h_{z_n}(\cdot)\}_n$ converges pointwise to $h_0(\cdot)$. The details of this example are left to the reader.

Notwithstanding, Proposition 3.10 yields the following striking result:

Corollary 3.11 (Gromov-compactifiability after renorming)**.** *For every normed space $(X, \|\cdot\|)$ there exists an equivalent norm $|||\cdot|||$ on X such that, the horofunction extension $\overline{(X, |||\cdot|||)}^h$ is a compactification of $(X, |||\cdot|||)$.*

Proof. Let $(X, \|\cdot\|)$ be a normed space. If $\dim(X) < \infty$, the result follows from Proposition 1.6. If $\dim(X) = \infty$, let Y be a closed hyperplane of X and $x \in X \setminus Y$. Then, $X = Y \oplus \mathbb{R}x$. Since X is linearly isomorphic to $Y \oplus_\infty \mathbb{R}x$, Proposition 3.10 (iii) finishes the proof. ■

Notice that the above corollary applies in particular to ℓ^1. Similarly to any normed space, this highly pathological space becomes Gromov-compactifiable under suitable renormings. Before we proceed, let us recall the following terminology:

Definition 3.12. *A property $(\mathcal{P})$ on Banach spaces is called a* 3*-space property (in short,* 3-SP*) if, for any Banach space X and any closed subspace $Y \subset X$ the following holds:*

If two of the spaces X, Y and X/Y satisfy $(\mathcal{P})$, then the third space also satisfies $(\mathcal{P})$.

Corollary 3.13. *The property of being Gromov-compactifiable is not stable neither under bi-Lipschitz homeomorphism nor subsets and it is not a 3-SP.*

Proof. The first assertion follows directly from Corollary 3.11. For the second assertion, it suffices to consider the space $X = \mathbb{R} \oplus_2 \ell^1$ (so that $\ell^1 \subset X$) and apply Proposition 3.10. Finally, to show that the property of being Gromov-compactifiable is not a 3-SP, consider $Y = \mathbb{R} \oplus_2 \{0\}$. Then $X/Y = \{0\} \oplus_2 \ell^1$. Therefore, X and Y are Gromov-compactifiable, but the space X/Y, being isometric to ℓ^1, is not. ■

Let us now elaborate on the statement of Theorem 3.2 (c). In the following example we show that the subspace F, evoked in the aforementioned statement, cannot be taken one-dimensional and needs to be of dimension at least 2. The question of whether the dimension of F needs to be arbitrarily large remains open.

Example 3.14 (Dimension of F in Theorem 3.2 (c))**.** Consider the space $X = \ell^1 \oplus_2 \ell^1$. Thanks to Proposition 3.10, X is Gromov-compactifiable, that is, the horofunction extension $\overline{X}^h$ is a compactification of X (and consequently, (c) of Theorem 3.2 holds).

We shall show that, for every one-dimensional subspace $F \subset X$ we have

$$d_H(S_F, S_X) = 2,$$

yielding that the subspace F given by Theorem 3.2 (c) satisfies in this case $\dim F \geq 2$.

Indeed, let $w \in X$ be a unit vector and set $F = \mathbb{R}w$. Let $w^1 \in \ell^1 \times \{0\}^{\mathbb{N}}$ and $w^2 \in \{0\}^{\mathbb{N}} \times \ell^1$ be such that $w = w^1 + w^2$. Then,

$$1 = \|w\|^2 = \|w^1\|^2 + \|w^2\|^2.$$

For sake of brevity, set $a := \|w^1\|$ and $b := \|w^2\|$. Denote by (e_n) and (f_n) the canonical bases of ℓ^1 for the first and second coordinate of X respectively. Then, for any $n \in \mathbb{N}$, $x_n := ae_n + bf_n \in X$ is a unit vector. Recalling that $S_F = \{w, -w\}$, we compute

$$\begin{aligned}
\|x_n \pm w\|^2 &= \|ae_n \pm w^1\|^2 + \|bf_n \pm w^2\|^2 \\
&= \left(|a \pm w^1_n| + \sum_{\substack{k=0 \\ k\neq n}}^{\infty} |w^1_k| \right)^2 + \left(|b \pm w^2_n| + \sum_{\substack{k=0 \\ k\neq n}}^{\infty} |w^2_k| \right)^2 \\
&= (|a \pm w^1_n| - |w^1_n| + a)^2 + (|b \pm w^2_n| - |w^2_n| + b)^2
\end{aligned}$$

Taking supremum on $n \in \mathbb{N}$, we deduce

$$\sup_{n\in\mathbb{N}} \min \left\{ \|x_n - w\|^2, \|x_n + w\|^2 \right\} \;\geq\; (2a)^2 + (2b)^2 = 4$$

and we conclude that $d_H(S_F, S_X) = 2$.

Let us start with the following result.

Corollary 3.15 (Isometric embedding)**.** *Every metric space (X, d) can be isometrically embedded in a Gromov-compactifiable space.*

Proof. It is well-known ([19, Theorem 1.6]) that every metric space (X, d) can be isometrically embedded into the Banach space $\ell^\infty(X)$ of all bounded real functions on X (equipped with the sup-norm) by means of the so-called Kuratowski-embedding. Namely, fixing a base point $x_0 \in X$, the map $x \mapsto \theta_x(\cdot)$, $x \in X$, where

$$\theta_x \equiv \{d(x, z) - d(x_0, z)\}_{z\in X} \in \ell^\infty(X)$$

is an isometry from X to $\ell^\infty(X)$. By Proposition 3.10 (iv) we deduce that $\ell^\infty(X)$ is Gromov-compactifiable and the conclusion follows. ∎

We shall now discuss two classical paradigms of canonical embeddings of a metric space X. The first one is the embedding to the so-called *Lipschitz-free* space $\mathcal{F}_{x_0}(X)$ (also known as *Arens-Eells* space or *Transportation cost* space). Let us provide a quick construction of $\mathcal{F}_{x_0}(X)$. (For a more detailed construction and classical properties we refer to [14].) Fix $x_0 \in X$ a base point and consider the Banach space $\mathrm{Lip}_{x_0}(X)$ of real-valued Lipschitz functions that vanish at x_0, equipped with the norm of the Lipschitz constant. Let $\delta : X \to \mathrm{Lip}_{x_0}(X)^*$ be the evaluation map defined by

$$\langle \delta(x), f \rangle = f(x), \text{ for all } x \in X, \ f \in \mathrm{Lip}_{x_0}.$$

It is known that δ is a (non-linear) isometry and that

$$\|\delta(x)\| = d(x, x_0), \quad \text{for all } x \in X. \tag{3.1}$$

The Lipschitz-free space $\mathcal{F}_{x_0}(X)$ is then defined as the closed linear span of $\delta(X)$ in $\mathrm{Lip}_0(X)^*$ (equipped with the restriction of the underlying norm), that is

$$\mathcal{F}_{x_0}(X) := \overline{\mathrm{span}}\,\{\delta(x) : \ x \in X\}.$$

It turns out that $\mathcal{F}_{x_0}(X)^* = \mathrm{Lip}_{x_0}(X)$. Furthermore, the isometric structure of $\mathcal{F}_{x_0}(X)$ is independent of the chosen base point x_0. This space is usually denoted by $\mathcal{F}(X) := \mathcal{F}_{x_0}(X)$, and its norm by $\|\cdot\|_{\mathcal{F}}$. In this framework, Theorem 3.4 (characterization of Gromov-compactifiability via non-octahedrality), clearly relates to the complete study about octahedrality in Lipschitz-free spaces that has been recently carried out by Procházka and Rueda-Zoca in [22].

The second paradigm is the *Wasserstein spaces*, which are classes of (probability) spaces that are associated with a given metric space X and relate to the *Optimal Transport* theory, see [10] for details. In particular, the 1-Wasserstein space $(P^1(X), W_1)$ of X, which consists of the Radon probability measures on X with finite first moment, offers another canonical isometrical embedding. The 1-Wasserstein distance of two elements (measures) $\mu, \nu \in P^1(X)$ is given by the formula (see [7, Theorem 4.1])

$$W_1(\mu,\nu) = \sup\left\{ \int_X f(x)d\mu(x) - \int_X f(x)d\nu(x) : \ f \in \mathrm{Lip}(X), \ \mathrm{Lip}(f) \le 1 \right\}.$$

In view of [7, Theorem 6.1], the set of probabilities with finite support (that is, convex combination of Dirac measures) is dense in $P^1(X)$. Therefore, $P^1(X)$ is isometrically isomorphic to $\overline{\mathrm{conv}}^{\mathcal{F}(X)}(\delta(X)) \subset \mathcal{F}(X)$.

We are now ready to state the following result, which asserts that (in contrast to Corollary 3.13) the property of being Gromov-compactifiable is inherited to X from either its Lipschitz-free space $\mathcal{F}(X)$ or its 1-Wasserstein space $P^1(X)$. In what follows, allowing a slight abuse of notation, for every $x \in X$, we shall denote by $\delta(x)$ both the element of (the vector space) $\mathcal{F}(X)$ and the Dirac measure in $P^1(X)$.

Proposition 3.16. *Let (X,d) be a metric space. Then*

(i). *X is Gromov-compactifiable when $\mathcal{F}(X)$ is Gromov-compactifiable.*

(ii). *X is Gromov-compactifiable when $P^1(X)$ is Gromov-compactifiable.*

Proof. If X is a finite metric space, then $\mathcal{F}(X)$ is finite dimensional and $P^1(X)$ is a polytope. In this case, the result follows easily from Proposition 1.6.

Let us now assume that the cardinality of X is infinite and, towards a contradiction, that X is not Gromov-compactifiable. Then by Theorem 2.1 (b), there exist $x_0 \in X$ and $r > 0$ such that for any compact set $K \subset X$ and any $\eta > 0$, there exists $z := z_{K,\eta} \in X \setminus \overline{B}(x_0, r)$ such that

$$d(z,w) > d(z,x_0) + d(x_0,w) - \eta, \text{ for all } w \in K.$$

In particular, for $0 < \eta < r$ and every finite set $A \subset X$, we set

$$z := z_{A,\eta}. \tag{3.2}$$

Moreover, for any choice $(\lambda_w)_{w\in A}$, we set

$$m_A := \sum_{w\in A} \lambda_w \, \delta(w) \ \big(\in \operatorname{span}(\delta(X)) \subset \mathcal{F}_{x_0}(X).\big)$$

Claim. There exists a 1-Lipschitz function $f : X \to \mathbb{R}$ such that

$$f(x_0) = 0, \quad \langle m_A, f\rangle = -\|m_A\|_{\mathcal{F}} \quad \text{and} \quad f(z) = d(z,x_0) - \eta. \tag{3.3}$$

Proof of the claim. Since $\mathcal{F}_{x_0}(X)^* = \mathrm{Lip}_{x_0}(X)$, there exists a 1-Lipschitz function $g : X \to \mathbb{R}$ with $g(x_0) = 0$ such that $\langle m_A, g\rangle = -\|m_A\|_{\mathcal{F}}$. Denote by g_1 the restriction of g to $A \cup \{x_0\} \subset X$ and consider the extension of g_1 to $A \cup \{x_0, z\}$ as follows:

$$g_2 : \ A \cup \{x_0, z\} \to \mathbb{R} \qquad \text{with } \ g_2|_{A\cup\{x_0\}} \equiv g_1 \quad \text{and} \ \ g_2(z) = d(x_0,z) - \eta$$

It is easy to check that g_2 is 1-Lipschitz, since for every $w \in A$ we have:

$$|g_2(z) - g_2(w)| \le |g_2(z)| + |g_2(w)| \le d(x_0,z) - \eta + d(w,x_0) < d(z,w).$$

We define f as any McShane extension of g_2 to X and the claim is proved.

Thanks to [7, Theorem 4.1], for the particular case where $\sum_{w\in A} \lambda_w = 1$ and $\lambda_w \ge 0$, we deduce

$$\langle m_A, f\rangle = -W_1\left(\delta(x_0), m_A\right) \qquad \text{and} \qquad f(z) = W_1(\delta(x_0), \delta(z)) - \eta. \tag{3.4}$$

(i). We shall use as base point the above point $x_0 \in X$ and prove that the Lipschitz-free space $\mathcal{F}_{x_0}(X)$ is not Gromov-compactifiable. Indeed, we are going to show that condition (c) of Theorem 3.2 fails. To this end, let F be a finite dimensional subspace of $\mathcal{F}_{x_0}(X)$ and $0 < \eta < r$. Let further $(\mu_k)_{k=1}^n$ be a finite η-net of the (compact) unit sphere

$S_F := F \cap S_{\mathcal{F}_{x_0}(X)}$ of F. By density of the subspace span$\,(\delta(X))$ in $\mathcal{F}_{x_0}(X)$, there exists $(m_k)_{k=1}^n \subset \text{span}(\delta(X))$, with $\|m_k\|_{\mathcal{F}} = 1$, such that

$$\|\mu_k - m_k\|_{\mathcal{F}} < \eta, \text{ for } k \in \{1, ..., n\}.$$

For every $k \in \{1, ..., n\}$, take $(x_{k,i})_{i=1}^{n_k} \subset X$ and $(\lambda_{k,i})_{i=1}^{n_k} \subset \mathbb{R}$ such that

$$m_k := \sum_{i=1}^{n_k} \lambda_{k,i}\delta(x_{k,i}).$$

Set $A_k := \bigcup_{i=1}^{n_k}\{x_{k,i}\}$ and $A = \bigcup_{k=1}^{k} A_k$. Take $z_A := z_{A,\eta}$ given in (3.2) and let f_k be a 1-Lipschitz function that satisfies (3.3), for the vector m_k and the point z_A.

Let $\mu \in S_F$. Then there is $k \geq 1$ such that $\|\mu - \mu_k\|_{\mathcal{F}} \leq \eta$. Considering the corresponding f_k and recalling by (3.1) that

$$\|\delta(z_A)\|_{\mathcal{F}} = d(z_A, x_0) > r \qquad \text{and} \qquad \langle f_k, m_k\rangle = -\|m_k\|_{\mathcal{F}} = -1$$

we deduce:

$$\begin{aligned}\Big\| \underbrace{\frac{\delta(z_A)}{d(x_0, z_A)}}_{S_{\mathcal{F}_{x_0}(X)}} - \underbrace{\mu}_{S_F} \Big\|_{\mathcal{F}} &\geq \Big\|\frac{\delta(z_A)}{d(x_0, z_A)} - \mu_k\Big\|_{\mathcal{F}} - \eta \geq \Big\|\frac{\delta(z_A)}{d(x_0, z_A)} - m_k\Big\|_{\mathcal{F}} - 2\eta \\ &\geq \langle f_k, \frac{\delta(z_A)}{d(x_0, z_A)} - m_k\rangle - 2\eta = \frac{f(z_A)}{d(x_0, z_A)} + \underbrace{\langle f_k, -m_k\rangle}_{=1} - 2\eta \\ &= \left(1 - \frac{\eta}{d(x_0, z_A)}\right) + 1 - 2\eta \geq 2 - \frac{\eta}{r} - 2\eta.\end{aligned}$$

Therefore, we obtain

$$d_H(S_F, S_{\mathcal{F}(X)}) \geq d\left(\frac{\delta(z_A)}{d(x_0, z_A)}, S_F\right) := \inf_{\mu\in S_F} \Big\|\frac{\delta(z_A)}{d(x_0, z_A)} - \mu\Big\|_{\mathcal{F}} \geq 2 - \eta\left(2 + \frac{1}{r}\right).$$

Since η is arbitrary, we eventually conclude that $d_H(S_F, S_{\mathcal{F}(X)}) = 2$.

(ii). We shall show that the 1-Wasserstein space $(P^1(X), W)$ is not Gromov-compactifiable. Following the same pattern of proof as in (i), we shall show that condition (b) of Theorem 2.1 fails for $P^1(X)$ at $\delta(x_0)$ and $r > 0$.

To this end, let $0 < \eta < r$ and $K \subset P^1(X)$ be a compact set. Let $(m_k)_{k=1}^n \in \text{conv}(\delta(X))$ be such that

$$\min_{k\geq 1} W_1(\mu, m_k) \leq \eta, \qquad \text{for any } \mu \in K.$$

Abusing slightly notation, we still write $m_k = \sum_{i=1}^{n_k} \lambda_{k,i}\delta(x_{k,i})$ and set $A_k = \bigcup_{i=1}^{n_k}\{x_{k,i}\}$ and $A = \bigcup_{k=1}^{n} A_k$ (as in the above proof). Consider $z_A := z_{A,\eta}$ as in (3.2). For any $\mu \in K$, fixing k such that $W_1(\mu, m_k) < \eta$ and considering f_k satisfying (3.4) with respect to m_k and z_A, we obtain

$$\begin{aligned} W_1(\delta(z_A), \mu) &\geq W_1(\delta(z_A), m_k) - \eta \geq \langle f_k, \delta(z_A) - m_k \rangle - \eta \\ &\underbrace{=}_{(3.4)} \big(W^1(\delta(z_A), \delta(x_0)) - \eta\big) + W^1(\delta(x_0), m_k) - \eta \\ &\geq W^1(\delta(z_A), \delta(x_0)) + W^1(\delta(x_0), \mu) - 3\eta. \end{aligned}$$

Since η is arbitrary, statement (b) of Theorem 2.1 cannot hold for $P^1(X)$ at $\delta(x_0)$.

The proof is complete. ■

Note that the converse of Proposition 3.16 (i) does not hold. For example, we can consider $X = \mathbb{R}$, which is Gromov-compactifiable. In this case, $\mathcal{F}(\mathbb{R})$ is isometrically isomorphic to $L^1(\mathbb{R})$, which is not Gromov-compactifiable (see Proposition 3.9). It would be interesting to know whether the converse of Proposition 3.16 (ii) holds true.

Acknowledgment. The authors thank the anonymous referee for careful reading and suggestions to improve the presentation of the manuscript. This work was initiated during a research visit of the second and the third author at TU Wien (April 2024). These authors thank TU Wien and the VADOR group for hospitality. This research was funded in whole or in part by the Austrian Science Fund (FWF) [DOI 10.55776/P36344N]. For open access purposes, the first author has applied a CC BY public copyright license to any author accepted manuscript version arising from this submission.

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Research supported by the FWF (Austrian Science Fund) grant DOI 10.55776/P-36344N.

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Research of M. I. Garrido and J. A. Jaramillo partially supported by Ministerio de Ciencia, Innovación y Universidades (Spain) grant PID2022-138758NB-I00.