Integration of Fenchel subdifferentials of epi-pointed functions.

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Abstract It is shown that in finite dimensions, Rockafellar's technique of integrating cyclically monotone operators applying to the Fenchel subdifferential of an epi-pointed function yields the closed convex hull of the function.

Key words Fenchel subdifferential, cyclically monotone operator, integration, epi-pointed function.

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Abbreviated title Fenchel subdifferential and integration

1 Introduction

By the term *integration of a multivalued operator* $T : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ we mean the problem of finding a lower semicontinuous (in short lsc) function f such that $T \subseteq \partial f$, where ∂f corresponds to some notion of subdifferential for the function f. This problem has recently attracted researchers interest, see for instance [3], [5], [6], [9] and references therein.

If we impose the further restriction that ∂f is the Fenchel subdifferential (see definition below), then a complete answer (even in infinite dimensions) to the aforementioned problem has been established by Rockafellar [7], with the introduction of the class of cyclically monotone operators. Indeed, as shown in [7] (see also [4]), every such operator T is included in the subdifferential ∂f of a lsc convex function f. In particular T coincides with ∂f if and only if it is maximal, and in such a case f is unique up to a constant.

Dealing with the above problem Rockafellar used a certain technique consisting on a formal construction of a lsc convex function f_T started from a given cyclically monotone operator T. The function f_T is further called the *convex integral* of T. Let us recall that Fenchel subdifferentials are particular cases of cyclically monotone operators. Consequently for every lsc function f with $\operatorname{dom} \partial f \neq \emptyset$, the convex integral $f_{\partial f}$ (also denoted $\widehat{f}$ in this paper) of its subdifferential ∂f defines naturally a lsc convex function minorizing f. If in particular f is convex, then the convex integral $\widehat{f}$ is equal to f up to a constant ([7]). In the general case, a natural question arises:

($\mathcal{Q}$) Given a lsc function f, is $\widehat{f}$ equal to the closed convex hull $\overline{\text{co}}\, f$ of f ?

This question has first been considered in [1, Proposition 2.6], where the authors provide a positive answer (in finite dimensions) for the class of *strongly coercive functions*, that is functions satisfying

$$\lim_{\|x\| \to \infty} \frac{f(x)}{\|x\|} = +\infty. \tag{1}$$

In this paper we improve the above result by establishing the same conclusion for the larger class of *epi-pointed functions* introduced in [2] (see definition below). Moreover, we shall give an easy example of a non epi-pointed function for which $(\mathcal{Q})$ is no more valid. However, for the one-dimensional case ($d = 1$), we shall show that $(\mathcal{Q})$ holds true for every lsc function defined on $\mathbb{R}$.

The paper is organized as follows. In the following section, we fix our notation and give some preliminaries concerning Fenchel duality and convex integration of the (Fenchel) subdifferential of a non convex function. The result of [1] for the class of strongly coercive functions is recalled and an example where the convex integration does not yield the closed convex hull of the function is illustrated. Finally in Section 3 we state and prove the main result of this article, concerning the class of epi-pointed functions.

2 Convex integration

Throughout this paper, we consider the Euclidean space $\mathbb{R}^d$ equipped with the usual scalar product $\langle \cdot, \cdot \rangle$. In the sequel, we denote by $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ a lsc function which is proper, that is $\operatorname{dom} f := \{x \in \mathbb{R}^d : f(x) \in \mathbb{R}\}$ is nonempty. We also denote by epi f the epigraph of f, that is the set $\{(x,t) \in \mathbb{R}^d \times \mathbb{R} : f(x) \leq t\}$. We recall that the second conjugate $\overline{\operatorname{co}}\, f$ (also denoted by f^{**}) of f is given by

$$\overline{\operatorname{co}}\, f(x) = \sup_{x^* \in \mathbb{R}^d} \{\langle x^*, x \rangle - f^*(x^*)\}, \tag{2}$$

where

$$f^*(x^*) = \sup_{x \in \mathbb{R}^d} \{\langle x^*, x \rangle - f(x)\}. \tag{3}$$

It is known that $\overline{\operatorname{co}}\, f$ is the greatest lsc convex function majorized by f, and that its epigraph coincides with the closed convex hull of the epigraph of f. By the term subdifferential, we shall always mean the Fenchel subdifferential ∂f defined for every $x \in \operatorname{dom} f$ as follows

$$\partial f(x) = \{x^* \in \mathbb{R}^d : f(y) \geq f(x) + \langle x^*, y - x \rangle, \forall y \in \mathbb{R}^d\}. \tag{4}$$

If $x \in \mathbb{R}^d \setminus \operatorname{dom} f$, we set $\partial f(x) = \emptyset$. Throughout this paper, the set

$$\operatorname{dom} \partial f := \{x \in \mathbb{R}^d : \partial f(x) \neq \emptyset\}$$

is supposed to be nonempty. Let further x_0 denote an arbitrary point of $\operatorname{dom} \partial f$. We call convex integral of ∂f the lsc convex function $\widehat{f} : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ defined for all $x \in \mathbb{R}^d$ by the formula

$$\widehat{f}(x) := f(x_0) + \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x_n^*, x - x_n \rangle \right\} \tag{5}$$

where the supremum is taken for all $n \geq 1$, all $x_1, x_2, ..., x_n$ in $\operatorname{dom} \partial f$ and all $x_0^* \in \partial f(x_0), x_1^* \in \partial f(x_1), \ldots, x_n^* \in \partial f(x_n)$. According to (4), we easily check that $\widehat{f} \leq f$ and consequently f is proper and

$$\widehat{f} \leq \overline{\mathrm{co}}\, f. \tag{6}$$

Rockafellar has shown ([8]) that if f is in particular convex, then the convex integral $\widehat{f}$ of ∂f is equal to f, that is

$$\widehat{f} = f. \tag{7}$$

In [1, Proposition 2.6] the authors generalized (7) to the nonconvex case by showing that if f is strongly coercive (that is f satisfies (1)), then (6) becomes

$$\widehat{f} = \overline{\mathrm{co}}\, f.$$

However the exact relation between $\widehat{f}$ and $\overline{\mathrm{co}}\, f$ for a function not satisfying (1) remains to discover. In particular, while in one-dimensional spaces we always have $\widehat{f} = \overline{\mathrm{co}}\, f$ (see Corollary 8), the following simple counterexample shows that this is not the case in general.

Example 1 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows:*

$$f(a,b) = \begin{cases} \exp(-a^2) + \frac{1}{2}b^2, & \textit{if } (a,b) \neq (0,0) \\ 0, & \textit{if } (a,b) = (0,0). \end{cases}$$

We easily check that

$$f^*(a,b) = \begin{cases} \frac{1}{2}b^2 & \textit{if } a = 0 \\ +\infty & \textit{if } a \neq 0 \end{cases}$$

and that

$$\overline{\mathrm{co}}\, f(a,b) = \frac{1}{2}b^2.$$

On the other hand, since

$$\partial f(a,b) = \begin{cases} \{0\} & \textit{if } (a,b) = (0,0) \\ \emptyset & \textit{if } (a,b) \neq (0,0) \end{cases}$$

formula (5) yields (for $x_0 = (0,0)$) that $\widehat{f}(x) = 0$ for all $x \in \mathbb{R}^2$. Hence $\widehat{f} \neq \overline{\mathrm{co}}\, f$.

Remark Modifying appropriately the function f around origin, we can obtain a continuous function $g : \mathbb{R}^2 \to \mathbb{R}$ such that $\widehat{g} \neq \overline{\mathrm{co}}\, g$.

Let us also remark that in the previous example we have

$$\mathrm{int}\,(\mathrm{dom}\ f^*) = \emptyset. \tag{8}$$

It will follow from the main theorem of Section 3 that (8) is in fact a necessary condition in order to obtain such examples.

3 Epi-pointed functions

The aim of this section is to establish the equality between the convex integral $\widehat{f}$ of ∂f and the closed convex hull $\overline{\mathrm{co}} f$ of f for the class of proper, lsc and epi-pointed functions defined in $\mathbb{R}^d$.

Let us recall the following definition ([2]).

Definition 2 *The function f is called epi-pointed if* $\mathrm{int}\,(\mathrm{dom}\ f^*) \neq \emptyset$.

It follows easily ([2, Proposition 4.5 (iv)]) that every strongly coercive function is epi-pointed. Note also that for every $\overline{x}^* \in \mathrm{int}\,(\mathrm{dom}\ f^*)$ we can always find $\overline{x} \in \mathbb{R}^d$ such that $f^*(\overline{x}^*) = \langle \overline{x}^*, \overline{x} \rangle - f(\overline{x})$ (that is the "sup" in (3) is attained). This obviously yields that $\overline{x}^* \in \partial f(x) \cap \mathrm{int}\,(\mathrm{dom}\ f^*)$. In particular, if f is epi-pointed the set $\mathrm{dom}\,\partial f$ is nonempty. If now x_0 is any point of $\mathrm{dom}\ \partial f$ we can consider the lsc convex function $\tilde{f}$ defined for all $x \in \mathbb{R}^d$ by

$$\tilde{f}(x) = f(x_0) + \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \ + \ \langle x_n^*, x - x_n \rangle \right\} \tag{9}$$

where the supremum is taken for all $n \geq 1$, all $x_1, x_2, \ldots, x_n$ in $\mathbb{R}^d$, all $x_0^* \in \partial f(x_0)$ and all

$$x_i^* \in \partial f(x_i) \cap \mathrm{int}\,(\mathrm{dom}\, f^*)$$

where $i \in \{1, \ldots, n\}$. Note that whenever f is epi-pointed, the set

$$\{x \in \mathbb{R}^d : \partial f(x) \cap \mathrm{int}\,(\mathrm{dom}\ f^*) \neq \emptyset\}$$

is nonempty, so that $\tilde{f}$ is proper. Comparing the formulas (5) and (9) we immediately conclude

$$\tilde{f} \leq \widehat{f}.$$

We shall show that if the function f is convex and epi-pointed, then f is equal to $\tilde{f}$ and so, in view of (7), the previous inequality becomes equality. This is the context of Proposition 4 below.

We shall first need the following lemma.

Lemma 3 *Suppose that f is lsc convex and epi-pointed. Then we have the inclusion*

$$\partial f^*(x^*) \subseteq \partial \tilde{f}^*(x^*), \ on\ \ \mathrm{int}\,(\mathrm{dom}\ f^*).$$

Proof A classical result (see [8]) states that for the lsc convex function f and all $x, x^* \in \mathbb{R}^d$ we have

$$x \in \partial f^*(x^*) \text{ if and only if } x^* \in \partial f(x).$$

Similarly, for the lsc convex function $\tilde{f}$

$$x \in \partial \tilde{f}^*(x^*) \text{ if and only if } x^* \in \partial \tilde{f}(x).$$

Let $x^* \in \operatorname{int}(\operatorname{dom} f^*)$ and $x \in \partial f^*(x^*)$. We shall show that $x \in \partial \tilde{f}^*(x^*)$. It follows that

$$x^* \in \partial f(x) \cap \operatorname{int}(\operatorname{dom} f^*). \tag{10}$$

For any $t < \tilde{f}(x)$, using the formula (9), we may choose $x_1, \ldots, x_n$ in $\mathbb{R}^d$, $x_0^* \in \partial f(x_0)$ and $x_1^* \in \partial f(x_1) \cap \operatorname{int}(\operatorname{dom} f^*), \ldots, x_n^* \in \partial f(x_n) \cap \operatorname{int}(\operatorname{dom} f^*)$ such that

$$t < f(x_0) + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x_n^*, x - x_n \rangle. \tag{11}$$

For any $y \in \mathbb{R}^d$, adding to both sides of (11) the quantity $\langle x^*, y - x \rangle$, we obtain

$$t + \langle x^*, y - x \rangle < f(x_0) + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle + \langle x_n^*, x - x_n \rangle + \langle x^*, y - x \rangle. \tag{12}$$

In view of (9), the right part of (12) is always less or equal to $\tilde{f}(y)$. Letting $t \to \tilde{f}(x)$ we infer

$$\tilde{f}(x) + \langle x^*, y - x \rangle \leq \tilde{f}(y)$$

which yields $x^* \in \partial \tilde{f}(x)$, or equivalently $x \in \partial \tilde{f}^*(x^*)$. □

Proposition 4 *If f is lsc convex and epi-pointed, then $\tilde{f} = f$.*

Proof Since the functions f^* and $\tilde{f}^*$ are proper, lsc and convex, we deduce from [8] and Lemma 3 that

$$f^* = \tilde{f}^* + k \text{ on } \operatorname{int}(\operatorname{dom} f^*) \tag{13}$$

for some constant $k \in \mathbb{R}$.

Let us now prove that the equality in (13) can be extended to all $\mathbb{R}^d$. According to [7, Corollary 7.3.4], it suffices to prove that the relative interiors of the convex sets $\operatorname{dom} f^*$ and $\operatorname{dom} \tilde{f}^*$ are equal, or equivalently (since $\operatorname{int}(\operatorname{dom} f^*)$ is nonempty) that

$$\operatorname{int}(\operatorname{dom} f^*) = \operatorname{int}(\operatorname{dom} \tilde{f}^*). \tag{14}$$

Let us now prove this last equality. Taking conjugates in both sides of the inequality $\tilde{f} \leq f$ we obtain $f^* \leq \tilde{f}^*$, hence in particular

$$\operatorname{dom} \tilde{f}^* \subseteq \operatorname{dom} f^*$$

and so

$$\operatorname{int}(\operatorname{dom} \tilde{f}^*) \subseteq \operatorname{int}(\operatorname{dom} f^*). \tag{15}$$

Conversely, let $x^* \in \operatorname{int}(\operatorname{dom} f^*)$. Since f^* is convex, we have $\partial f^*(x^*) \neq \emptyset$. By Lemma 3 we get $\partial \tilde{f}^*(x^*) \neq \emptyset$, yielding that $x^* \in \operatorname{dom} \partial \tilde{f}^*$. It follows that

$$\operatorname{int}(\operatorname{dom} f^*) \subseteq \operatorname{dom} \tilde{f}^*. \tag{16}$$

Combining (15) with (16), we conclude that equality (14) holds as desired. Hence, we obtain

$$f^* = \tilde{f}^* + k.$$

Taking conjugates, this last equality yields $f = \tilde{f} - k$. Since $f(x_0) = \tilde{f}(x_0)$ we conclude that $k = 0$ and thus $f = \tilde{f}$. $\square$

We shall finally need the following lemma.

Lemma 5 *Suppose that f is lsc and epi-pointed and set $g = \overline{\mathrm{co}}\, f$. Then, for any $x \in \mathrm{dom}\, \partial f$ and $x^* \in \partial g(x) \cap \mathrm{int}\,(\mathrm{dom}\, f^*)$ there exist $y_1, \ldots, y_p$ in $\mathbb{R}^d$ such that $x \in \mathrm{co}\,\{y_1, y_2, ..., y_p\}$ and*

$$x^* \in \bigcap_{i=1}^{p} \partial f(y_i)$$

Proof From [2, Theorem 4.6] we conclude that for any $x^* \in \partial g(x)$, there exist $y_1, \ldots, y_p$ in $\mathbb{R}^d$ and $w_1, \ldots, w_q$ in $\mathbb{R}^d \setminus \{0\}$ such that

$$x - \sum_{j=1}^{q} w_j \;\in\; \mathrm{co}\,\{y_1, y_2, ..., y_p\}$$

and

$$x^* \in \left[\bigcap_{i=1}^{p} \partial f(y_i)\right] \cap \left[\bigcap_{j=1}^{q} \partial f_\infty(w_j)\right], \tag{17}$$

where f_∞ is defined via the relation $\mathrm{epi}(f_\infty) = (\mathrm{epi}\, f)_\infty$ where

$$(\mathrm{epi}\, f)_\infty := \{d \in X : \exists \{x_n\}_{n\geq 1} \text{ in } \; \mathrm{epi}\, f, \exists \{t_n\} \searrow 0^+ \text{ with } d = \lim_{n\to+\infty} t_n x_n\}.$$

It suffices to show that for $x^* \in \mathrm{int}\,(\mathrm{dom}\, f^*)$, (17) yields $q = 0$. To this end suppose, towards a contradiction, that $q \neq 0$. Since the function f_∞ is sublinear, positively homogeneous and $f_\infty(0) = 0$ ([2] e.g.), it follows easily that for any $w_j \neq 0$ and any $x^* \in \partial f_\infty(w_j)$ we have $\langle x^*, w_j \rangle = f_\infty(w_j)$. Since $x^* \in \mathrm{int}\,(\mathrm{dom}\, f^*)$, we may find some $z^* \in \mathbb{R}^d$ (near x^*) such that $z^* \in \mathrm{int}\,(\mathrm{dom}\, f^*)$ and $\langle z^*, w_j \rangle > f_\infty(w_j)$. The latter yields easily that

$$z^* \notin \partial f_\infty(0). \tag{18}$$

On the other hand, since $z^* \in \mathrm{int}(\mathrm{dom}\, f^*) \subseteq \mathrm{dom}\, \partial f^*$ we conclude the existence of x in $\mathbb{R}^d$ such that $x \in \partial f^*(z^*)$, or equivalently

$$z^* \in \partial g(x). \tag{19}$$

Since $\partial g(x) \subseteq \partial f_\infty(0)$ ([2, Theorem 4.6]), relations (18) and (19) give the contradiction. $\square$

We are now ready to establish the main result of this section.

Theorem 6 *If f is lsc and epi-pointed then $\widehat{f} = \overline{\mathrm{co}}\, f$.*

Proof Set $g = \overline{\mathrm{co}}(f)$. Then g is lsc convex and $\mathrm{int}\,(\mathrm{dom}\ g^*) = \mathrm{int}\,(\mathrm{dom}\ f^*)$. In particular g is epi-pointed. Using Proposition 4 we conclude that

$$g(x) = g(x_0) + \sup \left\{ \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \ + \ \langle x_n^*, x - x_n \rangle \right\},$$

where the supremum is taken over all $n \geq 1$, all $x_1, \ldots, x_n$ in $\mathbb{R}^d$, all $x_0^* \in \partial g(x_0)$ and all

$$x_i^* \in \partial g(x_i) \cap \mathrm{int}\,(\mathrm{dom}\, f^*)$$

where $i \in \{1, \ldots, n\}$. Take any $x \in \mathbb{R}^d$ and any $t < g(x)$. Then there exist $x_1, \ldots, x_n$ in $\mathbb{R}^d$, $x_0^* \in \partial g(x_0)$ and $x_i^* \in \partial g(x_i) \cap \mathrm{int}\,(\mathrm{dom}\ f^*)$ (for $i = 1$ to n) such that

$$t < g(x_0) + \sum_{i=0}^{n-1} \langle x_i^*, x_{i+1} - x_i \rangle \ + \ \langle x_n^*, x - x_n \rangle. \tag{20}$$

Recalling that $x_0 \in \mathrm{dom}\, \partial f$, we easily check that $g(x_0) = f(x_0)$ and $\partial g(x_0) = \partial f(x_0)$. On the other hand, for all $i \in \{1, \ldots, n\}$ Lemma 5 guarantees the existence of points $y_i^1, \ldots, y_i^{p_i}$ in $\mathbb{R}^d$ such that $x_i \in \mathrm{co}\{y_i^1, y_i^2, \ldots, y_i^{p_i}\}$ and

$$x_i^* \in \bigcap_{j=1}^{p_i} \partial f(y_i^j).$$

We claim that for $i = 1$, there exists an index j_1 in $\{1, 2, ..., p_1\}$ such that

$$\langle x_0^*, x_1 - x_0 \rangle + \langle x_1^*, x_2 - x_1 \rangle \leq \langle x_0^*, y_1^{j_1} - x_0 \rangle + \langle x_1^*, x_2 - y_1^{j_1} \rangle.$$

Indeed, if this were not the case, then for every j we would have

$$\langle x_0^*, x_1 - x_0 \rangle + \langle x_1^*, x_2 - x_1 \rangle > \langle x_0^*, y_1^j - x_0 \rangle + \langle x_1^*, x_2 - y_1^j \rangle.$$

This yields a contradiction, since $x_1 \in \mathrm{co}\,\{y_1^1, \ldots, y_1^{p_1}\}$.

Proceeding like this for $i \geq 1$, we inductively replace all x_i's in (20) by $y_i^{j_i}$'s in a way that $x_i^* \in \partial f(y_i^{j_i})$, obtaining thus the formula

$$t < f(x_0) + \langle x_0^*, y_1^{j_1} - x_0 \rangle + \langle x_1^*, y_2^{j_2} - y_1^{j_1} \rangle + \cdots + \langle x_n^*, x - y_n^{j_n} \rangle.$$

Comparing with (5), we obtain $t < \widehat{f}(x)$. Letting $t \to g(x)$ we infer $g(x) = \overline{\mathrm{co}}\, f(x) \leq \widehat{f}(x)$, which finishes the proof in view of (6). $\square$

Corollary 7 *Suppose that f, h are proper lsc and epi-pointed functions. If $\partial f = \partial h$, then $\overline{\mathrm{co}}\, f$ and $\overline{\mathrm{co}}\, h$ are equal up to a constant.*

Proof For $x_0 \in \operatorname{dom} \partial f$ and $c = g(x_0) - f(x_0)$ we obviously have $\widehat{f} = \widehat{h} + c$, which in view of Theorem 6 yields $\overline{\mathrm{co}}\, f = \overline{\mathrm{co}}\, h + c$. $\square$

The class of proper, lsc and epi-pointed functions is not minimal in order to ensure the conclusion of Theorem 6. For example, every constant function f satisfies $\widehat{f} = \overline{\mathrm{co}}\, f = f$ and obviously dom $f^* = \{0\}$. (In fact, one can consider any lsc convex function f which is not epi-pointed.) Furthermore, the example of the function $f(x) = \min\{\|x\|, 1\}$ shows that the conclusion $\widehat{f} = \overline{\mathrm{co}}\, f$ might be true even in cases where f is non-convex and non epi-pointed at the same time. In particular, in one-dimensional spaces the following result is true.

Corollary 8 *If $d = 1$ (that is $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$) and $\operatorname{dom} \partial f \neq \emptyset$, then $\widehat{f} = \overline{\mathrm{co}}\, f$.*

Proof In view of Theorem 6, it suffices to consider only the case $\operatorname{int}(\operatorname{dom} f^*) = \emptyset$. Since f^* is convex (and $\operatorname{dom} \partial f \neq \emptyset$) it follows that dom $f^* = \{\alpha\}$, for some $\alpha \in \mathbb{R}$. We easily conclude from (2) that

$$\overline{\mathrm{co}}\, f(x) = \alpha x - f^*(\alpha), \tag{21}$$

for all $x \in \mathbb{R}$. On the other hand, for any $x_0 \in \operatorname{dom} \partial f$ we have $\partial f(x_0) = \{\alpha\}$ which yields in view of (3) and (4) that

$$f^*(\alpha) = \alpha x_0 - f(x_0). \tag{22}$$

Finally, it follows easily from relation (5) that

$$\widehat{f}(x) = f(x_0) + \alpha(x - x_0). \tag{23}$$

Relations (21), (22) and (23) yield directly $\widehat{f} = \overline{\mathrm{co}}\, f$. $\square$

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