Normal cones to sublevel sets: an axiomatic approach. Applications in quasiconvexity and pseudoconvexity

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Abstract. An axiomatic approach of normal operators to sublevel sets is given. Considering the Clarke-Rockafellar subdifferential (resp. quasiconvex functions), the definition given in [4] (resp. [5]) is recovered. Moreover, the results obtained in [4] are extended in this more general setting. Under mild assumptions, quasiconvex continuous functions are classified, establishing an equivalence relation between functions with the same normal operator. Applications in pseudoconvexity are also discussed.

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1 Introduction

The notion of a "normal cone to sublevel sets", i.e. a multivalued operator associating with every function f and every point x of its domain the normal cone to the sublevel set $S_{f(x)}$ has first been introduced and studied in [5], where the authors discussed continuity properties of this operator (or variants of it) when applied to quasiconvex functions. Subsequently, several authors used this notion (see [13], [10], [11] e.g.) for dealing with quasiconvex optimization problems.

In [4], a modification on the original definition ([5]) of the normal operator has been proposed, consisting in considering for every x the polar cone of the Clarke tangent cone of $S_{f(x)}$ at x. This new definition coincides with the previous one whenever the function f is quasiconvex, whereas it has the advantage to allow simple characterizations of various types of quasiconvexity in terms of corresponding types of quasimonotonicity of the normal operator.

In this work, following the lines of [4], we give an axiomatic formulation for the concept of normal operator, based on an abstract notion of subdifferential, see Section 2. Subsequently, we present some applications in quasiconvexity (Sections 3 and 5) and in pseudoconvexity (Section 4).

Throughout this paper, X will be a Banach space with dual X^*, and f a lower semicontinuous (lsc) function on X with values in $\mathbb{R} \cup \{+\infty\}$. For any $x \in X$ and any $x^* \in X^*$ we denote by $\langle x^*, x \rangle$ the value of the functional x^* at the point x. We also use the standard notation: $B_\delta(x)$ for the closed ball centered at x with radius $\delta > 0$, $\operatorname{dom} f := \{x \in X : f(x) \neq +\infty\}$ for the domain of the function f and $S_{f(x)} := \{x' \in X : f(x') \leq f(x)\}$ (resp. $S^-_{f(x)} = \{x' \in X : f(x') < f(x)\}$) for the sublevel and the strict sublevel sets of f. For $x, y \in X$ we set $[x, y] = \{tx + (1-t)y : 0 \leq t \leq 1\}$ and we define the segments $]x, y]$, $[x, y[$ and $]x, y[$ analogously.

2 Abstract subdifferential and normal operator

Let us first recall from [2] the definition of an abstract subdifferential.

Definition 1. We call *subdifferential operator*, any operator ∂ associating to any Banach space X, any lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ and any $x \in X$, a subset $\partial f(x)$ of X^*, and satisfying the following properties:

(P1) $\partial f(x) = \{x^* \in X^* \;:\; \langle x^*, y - x \rangle + f(x) \leq f(y), \quad \forall y \in X\}$, whenever f is convex;

(P2) $0 \in \partial f(x)$, whenever f attains a local minimum at $x \in \mathit{dom} f$;

(P3) $\partial(f + g)(x) \subset \partial f(x) + \partial g(x)$, whenever g is real-valued convex continuous, and *∂-differentiable at x*,

where g *∂-differentiable at x* means that $\partial g(x)$ and $\partial(-g)(x)$ are nonempty.

In the sequel, we shall assume in addition that

$$\partial \subset \partial^{\uparrow} \quad \text{or} \quad \partial \subset \partial^{D+}$$

where $\partial^{\uparrow}$ is the Clarke-Rockafellar and ∂^{D+} the upper Dini subdifferential.

Let us recall that the definitions:

$$\partial^{\uparrow} f(x) = \{x^* \in X^* : \langle x^*, d \rangle \leq f^{\uparrow}(x, d), \text{ for all } d \in X\}$$

where

$$f^{\uparrow}(x, d) = \sup_{\varepsilon > 0} \limsup_{\substack{t \searrow 0 \\ y \to_f x}} \inf_{d' \in B_\varepsilon(d)} \frac{1}{t} \left(f(y + td') - f(y) \right).$$

and

$$\partial^{D+} f(x) = \{x^* \in X^* : \langle x^*, d \rangle \leq f^{D+}(x, d), \text{ for all } d \in X\}$$

where

$$f^{D+}(x, d) = \limsup_{t \searrow 0^+} \frac{1}{t} \left(f(x + td) - f(x) \right).$$

It is recalled that $t \searrow 0^+$ indicates the fact that $t > 0$ and $t \to 0$, while $x \to_f x_o$ means that both $x \to x_o$ and $f(x) \to f(x_o)$.

We further recall from [2] the following definition.

Definition 2. A norm $\|.\|$ on X is said to be *∂-smooth* if the functions of the following form are ∂-differentiable:

$$x \mapsto \Delta_2(x) := \sum_n \mu_n \|x - v_n\|^2,$$

where $\mu_n \geq 0$, the series $\sum_n \mu_n$ is convergent, and the sequence (v_n) converges in X.

Let us also introduce the notion of an "abstract" normal cone, based on the subdifferential ∂.

Definition 3. Let ∂ be a subdifferential operator. For any closed subset C of X and any point $x \in X$ we associate the normal cone to C at the point x defined by

$$N_C(x) = \begin{cases} \partial \psi_C(x) & \text{if } x \in C \\ \emptyset & \text{otherwise} \end{cases}$$

where ψ_C denotes the indicator function of C (i.e. $\psi_C(x) = 0$ if $x \in C$ and $+\infty$ if $x \notin C$).

For all classical subdifferentials (Clarke, lower and upper Hadamard, lower and upper Dini, Frèchet, proximal...) the subset $N_C(x)$ is effectively a cone. Although this property will not be used in the sequel, to be in accordance with the term "normal cone" of the above definition, we can assume that the abstract subdifferential fulfills the following property:

For any function f, any $\lambda > 0$ and any $x \in X$, $\quad \partial(\lambda f)(x) = \lambda \partial f(x)$.

Whenever the subdifferential operator is the lower Hadamard subdifferential ∂^{H-}, the corresponding normal cone is the classical Bouligand normal cone defined as follows

$$NK_C(x) = \{x^* \in X^* \ : \ \langle x^*, d \rangle \leq 0, \ \forall d \in K_C(x)\} \tag{1}$$

with

$$K_C(x) = \{y = \lim_{k \to \infty} y_k \ : \ \exists\, t_k \searrow 0 \text{ with } x + t_k y_k \in C, \quad \forall\, k \in \mathbb{N}\}$$

On the other hand, if $\partial = \partial^{\uparrow}$, then we recover the Clarke normal cone

$$N_C^{\uparrow}(x) = \{x^* \in X^* \ : \ \langle x^*, d \rangle \leq 0, \ \forall d \in T_C(x)\} \tag{2}$$

with

$$d \in T_C(x) \Leftrightarrow \begin{cases} \forall \varepsilon > 0, \exists\, \delta > 0 \ \text{ such that} \\ \forall x' \in B_\delta(x) \cap C, \forall t \in]0, \delta[, \ (x' + tB_\varepsilon(d)) \cap C \neq \emptyset. \end{cases}$$

We are now in a position to define the normal operator associated with a function.

Definition 4. Let ∂ be a subdifferential operator. For any lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ we associate a multivalued operator $N_f : X \to 2^{X^*}$ - called normal operator - defined by

$$x \mapsto \begin{cases} N_{S_{f(x)}}(x) & \text{if } x \in \text{dom } f \\ \emptyset & \text{otherwise} \end{cases}$$

Remark: 1) In the particular case $\partial = \partial^{\uparrow}$, we recover the definition used in [4] (see relation (2)).

2) Based on the strict sublevel sets (i.e. $S_\lambda^- = \{x \in X \ : \ f(x) < \lambda\}$) an analogous concept of normal operator (called strict normal operator) has been considered in [4] (extending the original definition of [5]) :

$$\tilde{N}_f(x) = \begin{cases} \emptyset & \text{if } x \notin \text{dom } f \\ X^* & \text{if } x \in \text{Argmin} f \\ N_{cl(S^-_{f(x)})}(x) & \text{if } x \in cl(S^-_{f(x)}) \\ \{0\} & \text{otherwise} \end{cases}$$

Since, as showed in [4], the operator N_f is more appropriate than $\tilde{N}_f$ for the normal characterization of the different types of quasiconvexity, the use of (large) sublevel sets has been preferred for the purpose of this paper.

A natural question immediately arises concerning the relation between the multivalued operators N_f and ∂f and in particular, the possible equality between $N_f(x)$ and $cone\,(\partial f(x)) := \{tx^* \ : \ t \geq 0 \text{ and } x^* \in \partial f(x)\}$. This equality is not true in general. In fact several counterexamples have been given in [4] for the case $\partial = \partial^{\uparrow}$. In the following proposition we shed more light on this topic.

Let us recall that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called *quasiconvex* if its sublevel sets S_λ are convex subsets of X. Following [6], a locally Lipschitz function is said to be *regular* at a point x, if for any $d \in X$ the classical directional derivative $f'(x, d)$ exists and is equal to the Clarke directional derivative $f^\circ(x, d)$ defined as follows:

$$f^\circ(x, d) = \limsup_{t \searrow 0^+ \ y \to x} \frac{1}{t} \left(f(y + td) - f(y) \right)$$

Proposition 1. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lsc such that $0 \notin \partial f(X)$.*

i) If f is quasiconvex then, for any $x \in X$,

$$cone(\partial f(x)) \subset N_f(x).$$

ii) Let us suppose, in addition, that f is Lipschitz continuous and $\partial \subset \partial^{\uparrow}$. If f is quasiconvex or f is regular, then for any $x \in X$,

$$N_f(x) = cone(\partial f(x)).$$

Proof. For i) let us suppose, for a contradiction, that $x \in \mathrm{dom}\, f$ is such that $\partial f(x) \not\subset N_f(x) = NK_{S_f(x)}$. Hence there exists $y \in S_f(x)$ and $x^* \in \partial f(x)$ verifying $\langle x^*, y - x\rangle > 0$. Let $\delta > 0$ be such that $\langle x^*, u - x\rangle > 0$ for all $u \in B_\delta(y)$. Since f is quasiconvex, it follows (see [3] e.g.) that $f(u) \geq f(x)$ for all $u \in B_\delta(y)$. But, since y is an element of $S_f(x)$, y is a local minimum of f and therefore $0 \in \partial f(y)$ which contradicts the hypothesis.

ii) is a direct consequence of [6, Th. 2.4.7]. □

Remark 1. a) As proved in [14, Lemma 5.3], if ∂ is the Fréchet subdifferential, then assertion i) can be obtained without the assumption "$0 \notin \partial f(X)$".

b) In assertion ii) of the previous proposition, the Lipschitz assumption can not be dropped. Indeed, if we define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sqrt{x}$ if $x \geq 0$ and $f(x) = -\sqrt{-x}$ otherwise, then for any $x \neq 0$, $cone(\partial f(x)) = N_f(x)$, while for $x = 0$ we have $\partial f(0) = \emptyset$ and $N_f(0) = [0, +\infty[$.

3 Normal characterizations of quasiconvexity

In this section we establish 'normal' characterizations for quasiconvex and strictly (semistrictly) quasiconvex functions in terms of the abstract normal operator N_f. These characterizations have been derived in [4] in the particular case $\partial = \partial^\uparrow$.

Let us first recall the relevant definitions. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *semistrictly quasiconvex* if f is quasiconvex and for any x, $y \in \mathrm{dom}\, f$ we have

$$f(x) < f(y) \Longrightarrow f(z) < f(y),\ \forall\, z \in [x, y[.$$

Similarly, f is called *strictly quasiconvex*, if it is quasiconvex and for any x, $y \in \mathrm{dom}\, f$ and $z \in]x, y[$ we have

$$f(z) < \max\{f(x), f(y)\}.$$

For any subset K of X, let us also recall that a multivalued operator $T : X \to 2^{X^*}$ is called *quasimonotone* on K if for all $x, y \in K$ we have

$$\exists\, x^* \in T(x),\ \langle x^*, y - x\rangle > 0 \Rightarrow \forall\, y^* \in T(y)\ :\ \langle y^*, y - x\rangle \geq 0.$$

Following [8] T is called *cyclically quasimonotone* (on K), if for every $x_1, x_2, ..., x_n \in X$ (resp. $x_1, x_2, ..., x_n \in K$), there exists $i \in \{1, 2, ..., n\}$ such that

$$\langle x_i^*, x_{i+1} - x_i\rangle \leq 0, \forall x_i^* \in T(x_i)$$

(where $x_{n+1} := x_1$).

Furthermore ([7]), the operator T is called *semistrictly quasimonotone* on K, if T is quasimonotone on K and for any x, $y \in K$ we have

$$\exists\, x^* \in T(x),\ \langle x^*, y - x\rangle > 0 \Longrightarrow$$
$$\exists\, z \in]\frac{x+y}{2}, y[,\ \exists\, z^* \in T(z)\ :\ \langle z^*, y - z\rangle > 0.$$

Finally T is called *strictly quasimonotone* if T is quasimonotone and for any $x, y \in K$ we have

$$\exists\, z \in]x,y[,\ \exists\, z^* \in T(z) \ :\ \langle z^*, y-x\rangle \neq 0.$$

Let us now recall from [3] the following characterization.

Proposition 2. *Let X be a Banach space admitting a ∂-smooth renorm and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc function. Then f is quasiconvex iff ∂f is quasimonotone.*

For the forthcoming characterization we need the following lemmas:

Lemma 1. *Let C be a nonempty subset of X. The following statements are equivalent:*

i) *C is closed and convex.*
ii) *The indicator function ψ_C is convex and lsc.*
iii) *The indicator function ψ_C is quasiconvex and lsc.*

Proof. The proof is straightforward and will be omitted. ☐

Lemma 2. *For any lsc quasiconvex function f, and any $x \in dom(f)$ we have:*

$$N_f(x) = NK_{S_{f(x)}}(x)$$

Proof. For every $x \in dom f$, the set $C = S_{f(x)}$ is convex and closed, hence from Lemma 1 it follows that the function ψ_C is convex and lsc. Property (P1) of Definition 1 implies that $\partial\psi_C$ does not depend on the subdifferential operator. In particular $\partial\psi_C(x)$ coincides with the cones defined in (1) and (2) respectively. ☐

Theorem 1. *Let X be a Banach space admitting a ∂-smooth renorm and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc function. Consider the following statements:*

i) *f is a quasiconvex function.*

ii) $\exists\, x^* \in N_f(x),\ \langle x^*, y-x\rangle > 0 \Longrightarrow f(y) > f(x)$

iii) *N_f is a (cyclically) quasimonotone operator.*

Then we always have $i) \Rightarrow ii) \Rightarrow iii)$. Moreover if, either $\partial^{H-} \subset \partial$ and X admits a Gâteaux-smooth renorm or, $\partial \subset \partial^{D+}$ and f is radially continuous or, $\partial \subset \partial^{\uparrow}$ and f is continuous, then $iii) \Rightarrow i)$, hence all these three conditions are equivalent.

Proof. $i) \Rightarrow ii)$. Let us suppose that for some $x^* \in N_f(x)$ we have $\langle x^*, y-x\rangle > 0$. It follows from Lemma 2 that $N_f(x) = NK_{S_{f(x)}}(x)$. Consequently $y - x$ is not an element of $K_{S_{f(x)}}(x) = cl(\cup_{\lambda>0}\lambda(S_{f(x)} - \{x\}))$. Hence, in particular, y is not an element of $S_{f(x)}$, i.e. $f(x) < f(y)$.

ii) $\Rightarrow$ *iii)*. Take any finite family $\{x_1, \ldots, x_n\}$ of points of X and suppose that for $i \in \{1, \ldots, n\}$, there exists $x_i^* \in N_f(x_i)$ such that $\langle x_i^*, x_{i+1} - x_i \rangle > 0$ where $x_{n+1} = x_1$. A contradiction immediately occurs since *ii)* yields $f(x_1) < f(x_2) < \cdots < f(x_{n+1}) = f(x_1)$.

iii) $\Rightarrow$ *i)*. Let us suppose, to a contradiction, that f is not quasiconvex. Then from Lemma 1 it follows that for some $x_0 \in \operatorname{dom} f$, the function $\psi_{x_0} := \psi_{S_{f(x_0)}}$ is not quasiconvex.

If $\partial^{H-} \subset \partial$ (and X admits a Gâteaux-smooth renorm) then, in view of Proposition 2, its lower Hadamard subdifferential $\partial^{H-}\psi_{x_0}$ is not quasimonotone. Hence there exist x, $y \in \operatorname{dom} \psi_{x_0} = S_{f(x_0)}$, $x^* \in \partial^{H-}\psi_{x_0}(x)$ and $y^* \in \partial^{H-}\psi_{x_0}(y)$ satisfying $\langle x^*, y - x \rangle > 0$ and $\langle y^*, x - y \rangle > 0$. Note now that $S_{f(x)} \subseteq S_{f(x_0)}$, from which it follows that $\psi_{x_0}(\cdot) \leq \psi_x(\cdot)$. We can easily conclude that $\psi_{x_0}^{H^-}(x, d) \leq \psi_x^{H^-}(x, d)$ for all d in X, hence $\partial^{H^-}\psi_{x_0}(x) \subseteq \partial^{H^-}\psi_x(x)$. Hence $x^* \in N_f(x)$ and (similarly) $y^* \in N_f(y)$ and we obtain the desired contradiction.

In both other cases, using again Proposition 2, we conclude to the existence of x, $y \in \operatorname{dom} \psi_{x_0} = S_{f(x_0)}$, $x^* \in \partial\psi_{x_0}(x)$ and $y^* \in \partial\psi_{x_0}(y)$ satisfying $\langle x^*, y - x \rangle > 0$ and $\langle y^*, x - y \rangle > 0$.

Now we claim that $f(x) = f(y) = f(x_0)$.

[We obviously have $f(x) \leq f(x_0)$. Let us now suppose that $f(x) < f(x_0)$.

If $\partial \subset \partial^{D+}$, then from the radial continuity of f we may find some $\delta > 0$ such that $f(u) < f(x_0)$ for any element u in the segment $(x - \delta(y - x), x + \delta(y - x))$. Then it follows that the function ψ_{x_0} is constant on this segment, which is not compatible with the inequality $\langle x^*, y - x \rangle > 0$. Hence $f(x) = f(x_0)$ and for the same reasons $f(y) = f(x_0)$.

If now $\partial \subset \partial^{\uparrow}$ (and the function f is continuous), then we may take a $\delta > 0$ such that $f(u) < f(x_0)$ for all $u \in B_\delta(x)$, hence the function ψ_{x_0} is locally constant on x, which contradicts the fact that $\langle x^*, y - x \rangle > 0$. Again we conclude that $f(x) = f(x_0) = f(y)$. The claim is proved.]

Now the proof is complete. Indeed $\psi_{x_0} = \psi_x = \psi_y$. Hence, in both cases x^* is an element of $\partial\psi_{x_0}(x) = \partial\psi_x(x) = N_{S_{f(x)}}(x) = N_f(x)$ and y^* is an element of $N_f(y)$ thus furnishing a contradiction with the quasimonotonicity of N_f. $\square$

Using essentially the same proof as in [4] it is possible to obtain the following characterizations of semistrict and strict quasiconvexity in this more general framework. Let us thus state - without proof - these results.

Theorem 2. *Let X be a Banach space admitting a ∂-smooth renorm and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lsc and continuous on its domain $dom f$. Then the following statements are equivalent:*

i) f *is a semistrictly quasiconvex function.*

ii) $\exists\, x^* \in N_f(x) \;:\; \langle x^*, y-x\rangle > 0 \Longrightarrow f(y) > f(z),\ \forall\, z \in [x,y)$

iii) N_f *is a semistrictly quasimonotone operator on* $dom f$.

Theorem 3. *Let X be a Banach space admitting a ∂-smooth renorm and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc and continuous on $dom f$.*

Then f is strictly quasiconvex if and only if N_f is strictly quasimonotone on $dom f$.

4 Normal cones and pseudoconvexity.

In this section we shall discuss relations between normal operators and pseudoconvexity. In [1], a *differentiable* function f was called *pseudoconvex*, if for every $x, y \in dom(f)$ the inequality $\langle df(x), y-x\rangle \geq 0$ ensures $f(y) \geq f(x)$. The notion of pseudoconvexity was subsequently extended into non-smooth functions, based on the concept of subdifferential (see [12], [3]). Let us further give the definition of pseudoconvexity in an even more abstract setting.

Definition 5. Given an operator $T : X \to 2^{X^*}$, a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called T-pseudoconvex, if for any $x, y \in dom(f)$ and $x^* \in T(x)$, the inequality $\langle x^*, y-x\rangle \geq 0$ implies $f(y) \geq f(x)$.

In case $T := \partial f$, we recover the definition given in [12] (see also [9] for a summary).

Since Definition 5 of $N_f \setminus \{0\}$-pseudoconvexity and Theorem 1 *ii*) are very similar, one may wonder whether quasiconvexity and $N_f \setminus \{0\}$-pseudoconvexity differ. It is shown below (Proposition 3) that for some particular case these concepts coincide. However this is not the case in general, as shows the example of the function $f : \mathbb{R} \to \mathbb{R}$, with

$$f(x) = \begin{cases} 1, \text{ if } & |\, x \,| < 1 \\ 0, \text{ if } & |\, x \,| \geq 1 \end{cases} \tag{3}$$

The above function is lower semicontinuous and T-pseudoconvex (for $T = N_f \setminus \{0\}$), without being quasiconvex.

A more general example of a lsc function satisfying for all $x, y \in \mathrm{dom}\, f$ the property:

$$\forall x^* \in T(x),\ \langle x^*, y-x\rangle \geq 0 \Longrightarrow f(y) \geq f(z),\ \text{for all } z \in [x,y] \tag{4}$$

without being quasiconvex is given below. (Relation (4) was taken as definition for T-pseudoconvexity in [9]).

Example: Let us consider the lsc function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} 1 \text{ , if } & x < 0 \text{ and } y > 0 \\ 0 \text{ , if } & xy \geq 0 \\ -x \text{ , if } & x > 0,\, y < 0 \text{ and } -y \geq x \\ y \text{ , if } & x > 0,\, y < 0 \text{ and } -y \leq x. \end{cases} \tag{5}$$

It is easily seen that f is $N_f \setminus \{0\}$-pseudoconvex, provided that $\partial \subset \partial^{\uparrow}$. On the other hand, since

$$S_{f(0,0)} = \mathbb{R}^2 \setminus \{(x,y) \ : \ x < 0, \ y > 0\}$$

the function f is not quasiconvex.

Proposition 3. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc radially continuous function with convex domain. Then*

i) *f quasiconvex $\Rightarrow$ f $N_f \setminus \{0\}$-pseudoconvex.*

ii) *if, moreover, $X = \mathbb{R}^n$ and $\partial = \partial^{\uparrow}$ then f is quasiconvex iff f is $N_f(x) \setminus \{0\}$-pseudoconvex.*

Proof. $i)$ Let us assume that x, $y \in \operatorname{dom} f$ and $x^* \in N_f(x) \setminus \{0\}$ are such that $\langle x^*, y - x\rangle \geq 0$. Since $x^* \neq 0$, there exists $d \in X$ such that $\langle x^*, d\rangle > 0$. Then for $y_n = y + \frac{1}{n}d$ (with $n \in \mathbb{N}$) we have $\langle x^*, y_n - x\rangle > 0$ which implies, by $ii)$ of Theorem 1 that $f(y_n) > f(x)$. Since f is radially continuous this yields $f(y) \geq f(x)$ and f is $N_f \setminus \{0\}$-pseudoconvex.

$ii)$ To prove the converse implication, let us suppose that f is $N_f(x) \setminus \{0\}$-pseudoconvex and (towards a contradiction) z is an element of $]x, y[$ verifying

$$f(z) > \max[f(x), f(y)].$$

Since f is radially continuous, we may assume that $f(x) > f(y)$ and that there exists $\tilde{z} \in]z, y[$ such that $f(x) < f(\tilde{z}) < f(z)$. It is also no loss of generality in assuming that $f(u) > f(\tilde{z})$ for all $u \in]z, \tilde{z}]$. Thus $\tilde{z}$ is on the boundary of the closed subset $S_{f(\tilde{z})}$ and consequently $N_f(\tilde{z})$ contains a nonzero element $\tilde{z}^*$ (see [6] e.g.). On the other hand, since $f(\tilde{z}) > f(x)$, we have $\langle \alpha^*, x - \tilde{z}\rangle < 0$ for any $\alpha^* \in N_f(\tilde{z}) \setminus \{0\}$. In particular, $\langle \tilde{z}^*, y - \tilde{z}\rangle > 0$ and, according to the $N_f \setminus \{0\}$-pseudoconvexity, $f(y) > f(\tilde{z})$ which is a contradiction. □

We also recall ([8]) that an operator T is called *cyclically pseudomonotone*, if for every $x_1, x_2, ..., x_n \in X$, the following implication holds:

$$\begin{array}{l} \exists i \in \{1,2,...,n\}, \exists x_i^* \in T(x_i) : (x_i^*, x_{i+1} - x_i) > 0 \Longrightarrow \\ \exists j \in \{1,2,...,n\}, \forall x_j^* \in T(x_j) : (x_j^*, x_{j+1} - x_j) < 0 \end{array}$$

(where $x_{n+1} := x_1$).

Let us now state the following result, to be compared with Theorem 1.

Proposition 4. *Let X be a Banach space admitting a ∂-smooth renorm and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a continuous function. The following statements are equivalent:*

i) *f is quasiconvex.*

ii) *$N_f \setminus \{0\}$ is (cyclically) pseudomonotone.*

Proof. *i)* $\Rightarrow$ *ii)*. Set $T(x) = N_f(x) \setminus \{0\}$ for all $x \in X$. Let any finite subset $\{x_1, x_2, ..., x_n\}$ of X and suppose (for a contradiction) that $\langle x_1^*, x_2 - x_1 \rangle > 0$ whereas for all $j \geq 2$, and all $x_j^* \in T(x_j)$, $\langle x_j^*, x_{j+1} - x_j \rangle \geq 0$ (where $x_{n+1} := x_1$). Since $x_j^* \neq 0$, using the same arguments as in part *i)* of the previous proof, we obtain $f(x_{j+1}) \geq f(x_j)$, for $j \geq 2$. On the other hand, since $\langle x_1^*, x_2 - x_1 \rangle > 0$ we infer by Theorem 1 *ii)* that $f(x_2) > f(x_1)$. The contradiction follows easily, since $x_{n+1} := x_1$. Hence T is cyclically pseudomonotone.

ii) $\Rightarrow$ *i)*. This implication follows from Theorem 1 (*iii)* $\Rightarrow$ *i)*), since the pseudomonotonicity of $N_f \setminus \{0\}$ obviously implies the quasimonotonicity of N_f. □

It is well known (see [8] e.g.) that every ∂f-pseudoconvex lsc function is quasiconvex. Combining with Proposition 3 *i)* and proposition 1 *i)* we thus recover easily the following known result:

Corollary 1. *Suppose that f is continuous and $0 \notin \partial f(X)$. Then*
f is quasiconvex $\Longleftrightarrow$ f is ∂-pseudoconvex

5 Normally equivalent functions

As observed in [4], two functions with the same normal operator may differ by more than an additive constant. Nevertheless, using the previous definition of T-pseudoconvexity (with $T = N_f \setminus \{0\}$), it is possible to characterize, under certain regularity assumptions, the set of quasiconvex functions having the same normal operator as a given quasiconvex function. This is the aim of Theorem 4.

Let us first define an equivalent relation on the set of all real-valued functions on X as follows:

$$f \sim g \Leftrightarrow N_f(x) = N_g(x), \quad \forall x \in X.$$

Remark: It follows directly from the definition that $f \sim \varphi \circ f$ for every $f : X \to \mathbb{R}$ and every strictly increasing function $\varphi : \mathbb{R} \to \mathbb{R}$, since the functions f and $\varphi \circ f$ have the same sublevel sets.

We now denote by $\mathcal{C}$ the class of continuous quasiconvex functions $f : X \to \mathbb{R}$ satisfying the following two regularity conditions:

(a) every local minimum is a global minimum

(b) the subset

$$Argmin\ f := \{x \in X : f(x) = \inf_X f\}$$

is included in a closed hyperplane of X.

Let us remark that assumption (a) can be rewritten as follows:

(a') For every $\lambda \in f(X)$, $\lambda > \inf_X f$: $cl(S_\lambda^-) = S_\lambda$

and that, in finite dimensional spaces, (b) is equivalent to

(b') the subset $Argmin\ f$ has an empty interior.

Hypothesis (a) has been used in [5] in order to obtain continuity results for the normal operator.

In the following theorem, we characterize the equivalent class, denoted by $\bar{f}$, of a given function f in $\mathcal{C}$.

Theorem 4. *The equivalent class $\bar{f}$ of a given function f in $\mathcal{C}$ is the set of all $N_f \setminus \{0\}$-pseudoconvex functions, that is*

$$\bar{f} = \{g \in \mathcal{C}\ :\ \exists x^* \in N_f(x) \setminus \{0\}\ :\ \langle x^*, y - x\rangle \geq 0 \Longrightarrow g(y) \geq g(x)\}.$$

Another way to express this result is to say that a function g of $\mathcal{C}$ has the same normal operator as a given function f of $\mathcal{C}$ if, and only if, g is $N_f \setminus \{0\}$-pseudoconvex.

Proof. Let us denote by $\mathcal{C}_f$ the subset of $\mathcal{C}$ defined by

$$\mathcal{C}_f = \{g \in \mathcal{C}\ :\ \exists x^* \in N_f(x) \setminus \{0\}\ :\ \langle x^*, y - x\rangle \geq 0 \Longrightarrow g(y) \geq g(x)\}.$$

(i) Let us first show $\bar{f} \subseteq \mathcal{C}_f$:

Suppose that $g \in \bar{f}$ and let x, $y \in X$ and $x^* \in N_f(x) \setminus \{0\} = N_g(x) \setminus \{0\}$ be such that

$$\langle x^*, y - x\rangle \geq 0. \tag{6}$$

If the inequality (6) is strict, then from Theorem 1 we conclude $g(y) > g(x)$.

In case where equality holds in (6), there exists a sequence $(y_n)_n \subset X$ converging to y such that $\langle x^*, y_n - x\rangle > 0$, for any $n \in \mathbb{N}$. It follows $g(y_n) \geq g(x)$, which together with the upper semicontinuity of g yields $g(y) \geq g(x)$.

(ii) We shall now show $\mathcal{C}_f \subseteq \bar{f}$:

Let any $g \in \mathcal{C}_f$.

<u>*Step 1:*</u> $N_f(x) \subseteq N_g(x)$, for all $x \in X$.

Assume, for a contradiction, that there exists $x \in X$ and $x^* \in N_f(x)$ such that $x^* \notin N_g(x)$.

Claim: $x \in Argmin\ g$

[Indeed, if x is not an element of $Argmin\ g$ then, using assumption (a') and the fact that x^* is not an element of $N_g(x)$, we immediately obtain the existence of a point y of $S^-_{g(x)}$ satisfying $\langle x^*, y - x\rangle \geq 0$. A contradiction occurs since the definition of $\mathcal{C}_f$ now yields $g(y) \geq g(x)$. The claim is proved.]

Since $x^* \notin N_g(x)$, there exists $\bar{y} \in S_{g(x)} = Argmin\ g$ such that

$$\langle x^*, \bar{y} - x\rangle > 0. \tag{7}$$

Obviously

$$g(x) = g(\bar{y}) = \min g \tag{8}$$

On the other hand, x^* is an element of $N_f(x)$ and therefore, (7) implies, $f(\bar{y}) > f(x)$.

Pick now any λ in $]f(x), f(\bar{y})[$. Since f is continuous, there exists $\varepsilon > 0$ such that

$$B_\varepsilon(x) \subset S^-_\lambda(f). \tag{9}$$

Due to the closedness of $S_\lambda(f)$, one can find $t \in]0,1[$ such that

$$C_t \cap S_\lambda(f) = \emptyset \tag{10}$$

where $C_t = \{t\bar{y} + (1-t)u\ ;\ u \in B_\varepsilon(x)\}$. Since $int(C_t) \neq \emptyset$, assumption (b) implies the existence of a point $\tilde{x} \in B_\varepsilon(x)$ such that for $\tilde{y} = t\bar{y} + (1-t)\tilde{x}$ we have:

$$g(\tilde{y}) > g(\bar{y}) = g(x) \tag{11}$$

Thanks to (10), it is no loss of generality to assume that $f(z) > f(\tilde{x})$ for all z in $]\tilde{x}, \bar{y}]$. Applying thus a separation argument to the disjoint convex sets $]\tilde{x}, \bar{y}]$ and $S_{f(\tilde{x})}(f)$, we conclude that there exists $\tilde{x}^* \in N_f(\tilde{x}) \setminus \{0\}$ such that $\langle \tilde{x}^*, \bar{y} - \tilde{x}\rangle \geq 0$.

The definition of $\mathcal{C}_f$ now yields $g(\bar{y}) \geq g(\tilde{x})$. The contradiction is obtained, since, using (8) with the quasiconvexity of g we get $g(\tilde{y}) = g(\bar{y})$, which is not compatible with (11). Hence $N_f(x) \subseteq N_g(x)$, for all $x \in X$.

Step 2: $N_g(x) \subset N_f(x)$, for all $x \in X$.

We shall also proceed by contradiction. So let us suppose that there exist $x \in X$ and $x^* \in N_g(x)$ such that x^* is not an element of $N_f(x)$. This implies the existence of a point y of $S_{f(x)}(f)$ which is not in $S_{g(x)}(g)$, i.e. $g(y) > g(x)$.

Case 1: The interior of $S_{f(x)}(f)$ is nonempty.

In this case we claim that there exists $\bar{z}$ such that $f(\bar{z}) < f(x)$ and $g(\bar{z}) > g(x)$.

Indeed if $f(y) < f(x)$, then take $\bar{z} = y$. Otherwise we have $f(x) = f(y)$, and thanks to hypothesis (a') there exists a sequence $\{y_n\}_{n\geq 1}$ in $S^-_{f(x)}(f)$ converging to y. Since g is continuous and $g(y) > g(x)$, the claim follows for $\bar{z} = y_n$ and n sufficiently large.

Now one can separate (in a large sense) the subsets $S_{f(\bar{z})}(f)$ and $\{x\}$. Hence there exists $\bar{z}^* \in N_f(\bar{z}) \setminus \{0\}$ such that

$$\langle \bar{z}^*, x - \bar{z}\rangle \geq 0.$$

This immediately implies, from the definition of $\mathcal{C}_f$, that $g(\bar{z}) \leq g(x)$ which is impossible.

Case 2. The set $S_{f(x)}(f)$ has an empty interior.

In this case we have $f(x) = f(y) = \min f$. We shall conclude again to a contradiction. Indeed, by hypothesis (b) there exists $\alpha^* \in X^* \setminus \{0\}$ such that

$$Argmin\ f \subseteq H_{\alpha^*} = \{u \in X \ : \ \langle \alpha^*, u - y \rangle = 0\}$$

Thus $\alpha^* \in N_f(y) \setminus \{0\}$, hence according to the definition of $\mathcal{C}_f$, $g(x) \geq g(y)$ which is impossible.

Consequently N_f coincides with N_g and the proof is complete. □

Example: If $X = \mathbb{R}$, the class $\mathcal{C}$ consists of the equivalent classes determined by the functions $\bar{f}_1(x) = x$, $\bar{f}_2(x) = -x$ and $\bar{f}_{3,\alpha}(x) = |x - \alpha|$ (for $\alpha \in \mathbb{R}$). For example, the function defined in Remark 1 is an element of $\bar{f}_1$.

Remarks: 1. Two equivalent functions $f, g \in \mathcal{C}$ do not necessarily have the same family of sublevel sets. Consider for instance the functions $f(x) = |x|$ and $g(x) = \max\{x, -2x\}$. Note that both functions belong to the class defined by $\bar{f}_{3,0}$ (see the previous example).

2. It is possible to consider quasiconvex functions taking the value $+\infty$. In this case one can obtain a result similar to Theorem 4 under the assumption that all functions have the same domain. Without this assumption, the fore mentioned result is not true, as can be shown by easy counterexamples.

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